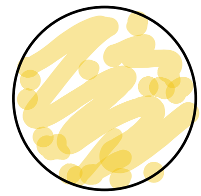
The ellipsoid method

Ellipsoids

Consider the unit ball in $\mathbb{R}^n$ centered at 0.

$$B(0,1) = \{ x \in \mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_n^2 \leq 1 \}$$
$$= \{ x \in \mathbb{R}^n : x^T x \leq 1 \}$$

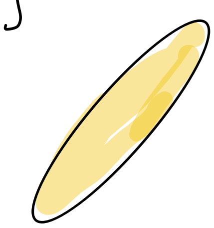


Let $A \in \mathbb{R}^n$ be a non-singular matrix, $c \in \mathbb{R}^n$

The image of the ball under this transformation

$$x \longmapsto Ax + c$$

is $\{ y : (y-c)^T \underbrace{(A^{-1})^T A^{-1}}_{\text{symmetric, positive definite}} (y-c) \leq 1 \}$



An ellipsoid centered at c is a set of the form

$$\mathrm{ell}(c, D) = \{ x \in \mathbb{R}^n : (x-c)^T D^{-1} (x-c) \leq 1 \}$$

D positive definite $\Leftrightarrow$ $\begin{cases} D \text{ has an orthonormal basis of eigenvectors} \\ \text{All eigen values of } D \text{ are positive.} \end{cases}$

The outline

$A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^n$

Assume: (i) The polyhedron $P := \{x : Ax \leq b\}$ is bounded and full-dimensional.

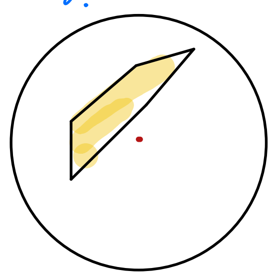
(ii) Calculations can be done precisely.

Let $\nu = 4n^2\varphi$, where φ is the maximum row size of the matrix $[A \mid b]$.

Fact: Each vertex of P has size at most ν.

Initial radius: $R = 2^{\nu}$

$P \subseteq B(0, R)$



Khachian: Determine a sequence of ellipsoids

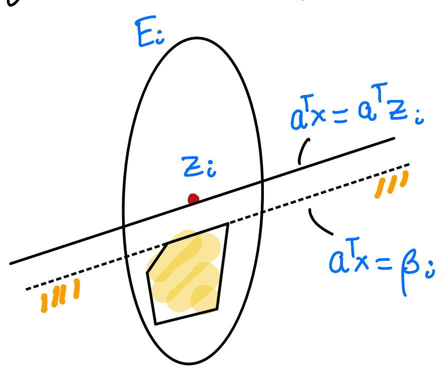
$$E_0, E_1, E_2, \dots, E_i, \dots \quad \text{s.t.} \quad P \subseteq E_i$$

$$B(0, R) = E(\underbrace{z_0}_{=\,0}, \underbrace{D_0}_{=\,R^2 \cdot I}) \qquad E_i = E(z_i, D_i)$$

Iteration: Suppose z_i and D_i have been found such that

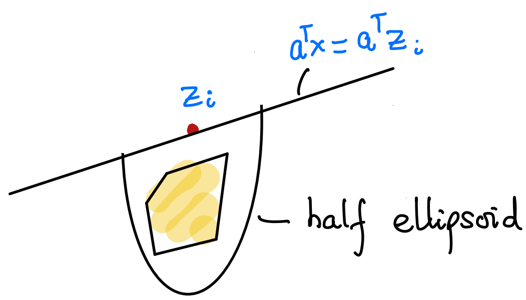
$$P \subseteq E(z_i, D_i)$$

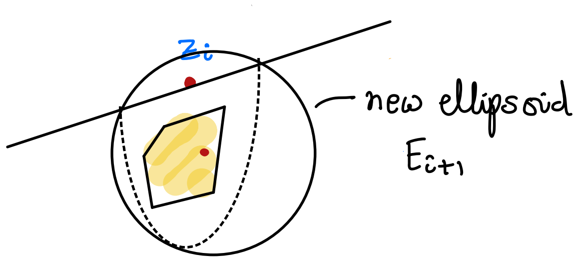
- If $z_i \in P$, we have found a feasible solution. STOP.
- If $z_i \notin P$, then it violates an inequality of the form $a^Tx \leq \beta$ in $Ax \leq b$.



Locate such an inequality and consider

$$E_i \cap \{x : a^Tx \leq \underbrace{a^Tz_i}_{\text{scalar}}\}$$





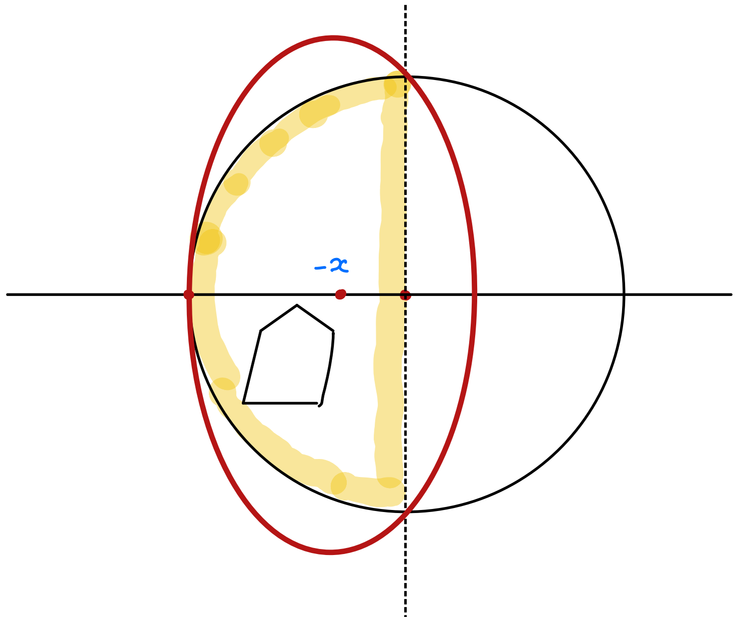
$$\left(1 - \frac{1}{2n+2}\right)$$

CLAIM 1: $\dfrac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-1/(2n+2)}$

CLAIM 2: $\mathrm{vol}(P) \geq 2^{-2n\nu}$

Proof of CLAIM 1: Consider the special case.

$$D_{new} = \begin{pmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{pmatrix}$$



$$\lambda_1^2 = (1-x)^2$$

$$\frac{x^2}{(1-x)^2} + \frac{1}{\lambda_2^2} = 1$$

$$\Downarrow$$

$$\lambda_2^2 = \frac{(1-x)^2}{1-2x}$$

$$\frac{vol(new)}{vol(old)} = \lambda_1 \lambda_2 \dots \lambda_n$$

$$= \frac{(1-x)^n}{(1-2x)^{(n-1)/2}}$$

to minimize this choose

$$x = \frac{1}{n+1}.$$

$$\frac{vol(new)}{vol(old)} = \frac{(n/n+1)^n}{\left((n-1)/(n+1)\right)^{(n-1)/2}}$$

$$= \left(\frac{n}{n+1}\right)\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}}$$

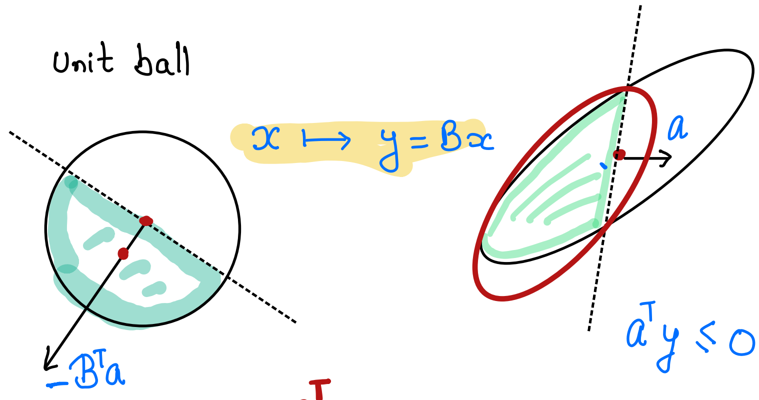
$$\leq \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}} \leq \exp\left(\frac{-1}{n+1}\right)\exp\left(\frac{n-1}{2(n+1)}\right)$$

$$= \exp\left(\frac{-1}{2(n+1)}\right)$$

In general,

$$z_{new} = z - \frac{1}{n+1} \frac{Da}{\sqrt{a^T D a}}, \quad D_{new} = \frac{n^2}{n^2-1}\left(D - \frac{2}{n+1} \frac{D a a^T D}{a^T D a}\right)$$

How?



$a^T B x \leq 0$

i.e., $(B^T a)^T x \leq 0$

$$\frac{B^T a}{\sqrt{a^T B B^T a}}$$

$a^T y \leq 0$

$y = Bx$

$$y^T \underbrace{(B^{-1})^T B^{-1}}_{D^{-1}} y \leq 1$$

$D = BB^T$

$$z_{new} = -\frac{1}{n+1} \frac{B^T a}{\sqrt{a^T B \cdot B^T a}}$$

$$= -\frac{1}{n+1} \frac{B^T a}{\sqrt{a^T D a}} \longmapsto -\frac{1}{n+1} \frac{B B^T a}{\sqrt{a^T D a}}$$

$$= -\frac{1}{n+1} \frac{Da}{\sqrt{a^T D a}}$$

Check that a similar computation yields

$$D_{new} = \frac{n^2}{n^2-1}\left(D - \frac{2}{n+1} \frac{D a a^T D}{a^T D a}\right)$$

Assumptions: (i) The polyhedron $P := \{x : Ax \le b\}$ is bounded and full-dimensional.

$A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^{n}$

(ii) Calculations can be done precisely.

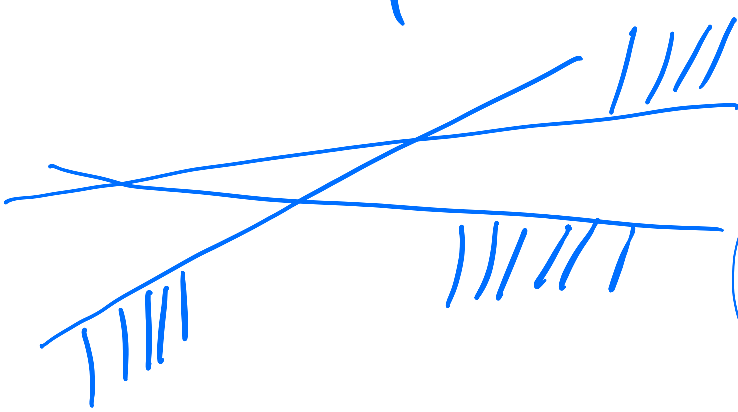
$$Ax \le b$$
$$\tilde{A}\tilde{x} = b$$
$$x \ge 0$$

We may assume all coordinates of x are at most ν.

$$-R \le x_i \le R$$

$$a^T x \le \beta + \varepsilon$$

$$Ax \le b + \begin{pmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix}$$



$$\boxed{y^T A \le y^T b}$$

$$\boxed{0 \le -1}$$

$$0 \le \underbrace{-1 + y^T \begin{pmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}}$$

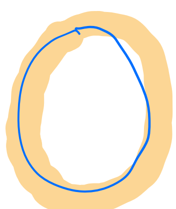
Approximations

Theorem (Approximating Ellipsoids)

$E = \mathrm{ell}(z, D)$, eigenvalues of $D \in [s^2, S^2]$.

$\varepsilon > 0$, $p \geq 3n + |\log_2 S| + 2|\log_2 s| + |\log_2 \varepsilon|$

$(\tilde{z}, \tilde{D}) = (z, D)$ rounded to p bits of the fractional part (keeping $\tilde{D}$ symmetric)



(i) eigenvalues$(\tilde{D}) \in \left[\frac{1}{2}s^2, 2S^2\right]$

(ii) $E \subseteq B(\tilde{E}, \varepsilon)$

(iii) $\mathrm{vol}(\tilde{E}) / \mathrm{vol}(E) \leq 1 + \varepsilon$

Theorem: $E = \mathrm{ell}(z, D)$, eigenvalues $\in [s^2, 2S^2]$.

H an affine space containing z; $\varepsilon \geq 0$.

$$B(E, \varepsilon) \cap H \subseteq B(E \cap H, \varepsilon S / s)$$