The Lovász theta function

Example: $G=(V,E)$, say $V=[n]=\{1,2,\dots,n\}$

For an edge $\{i,j\}\in E$, let $u_e = \begin{pmatrix}0\\0\\1\\0\\0\\1\\0\end{pmatrix}\begin{matrix}\\\\\leftarrow i\\\\\\\leftarrow j\\\\\end{matrix}$.

$$\boxed{\beta(G) = \min_{\substack{\omega: V\to\mathbb{R}\\ \omega\cdot\mathbb{1}=1}} \max_{e\in E} \frac{1}{|\langle w, u_e\rangle|}}$$

- Note that $\beta(G) = \min_{\substack{\omega: V\to\mathbb{R}\\ \omega\cdot\mathbb{1}=1}} \dfrac{1}{\min_{e\in E} |\langle w, u_e\rangle|} = \dfrac{1}{\max_{\substack{\omega: V\to\mathbb{R}\\ \omega\cdot\mathbb{1}=1}} \min_{e\in E} |\langle \omega, u_e\rangle|}$

So, this definition asks us to:

"Allocate a total amount of 1 unit among the vertices such that every edge receives a large amount."

$\beta(G)$ = the reciprocal of this large amount that every edge is guaranteed to receive.

- This is, in fact, a disguised version of the linear program for the fractional vertex cover problem. (Check!)

Using the same idea with ℓ_2 budgets, we obtain a formulation for independent sets.

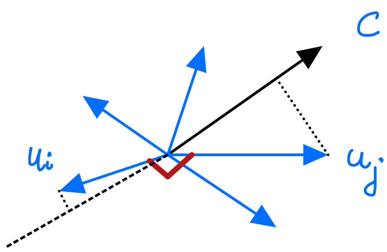
$$\Theta(G) = \min_{\substack{c \\ \|c\|=1}} \max_{i\in[n]} \frac{1}{\langle c, u_i\rangle^2}$$

The minimum exists! See the textbook.

each vertex is assigned a unit vector → $u_1, u_2, \dots, u_n$

$\|u_i\| = 1$

$u_i \perp u_j$ if $\{i,j\} \notin E$ ← non-adjacent vertices take up disjoint portions of the budget



i and j are non-adjacent

S independent

$\Downarrow$

$1 = \|c\|^2 \geq \sum_{i\in S} |\langle u_i, c\rangle|^2 \geq |S| / \Theta(G)$

<u>Theorem</u>: $\alpha(G) \leq \Theta(G)$

size of the largest independent set

Lovász's theta function

For all G, $\alpha(G) = \omega(\bar{G}) \leq \chi(\bar{G})$

clique number

chromatic number

Both $\Theta(G)$ and $\chi(\bar{G})$ are upper bounds for $\alpha(G)$.

Are they related?

Complete graph K_n : $\quad c = u_i = (1) \quad n=1$

$$1 = \alpha(K_n) \leq \Theta(K_n) \leq 1 \implies \Theta(K_n) = 1.$$

Empty graph $\overline{K}_n$:

$$u_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow i \quad \text{for } i = 1, 2, \dots, n, \qquad c = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle u_i, c \rangle = \frac{1}{\sqrt{n}}$$

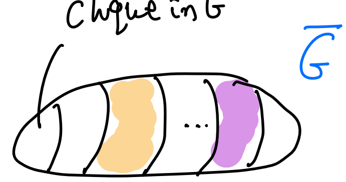
$$n \leq \alpha(\overline{K}_n) \leq \Theta(\overline{K}_n) = n$$

shown above

$$\boxed{\text{Sandwich theorem: } \alpha(G) = \omega(\overline{G}) \leq \Theta(G) \leq \chi(\overline{G})}$$

Theorem: $\Theta(G) \leq \chi(\overline{G})$

Suppose the colours are $1, 2, \dots, k$.



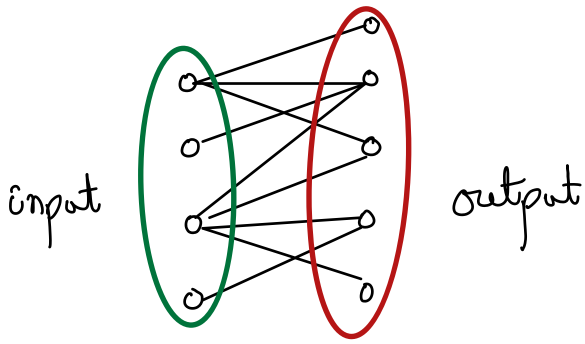
Each colour class is a clique in G.

Set $u_v = e_i$ for vertices v with colour i.

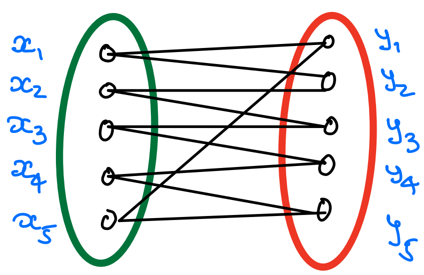
Set $c = \begin{pmatrix} 1/\sqrt{k} \\ \vdots \\ 1/\sqrt{k} \end{pmatrix}$. $\quad \Rightarrow \quad \Theta(G) \leq k = \chi(\overline{G})$

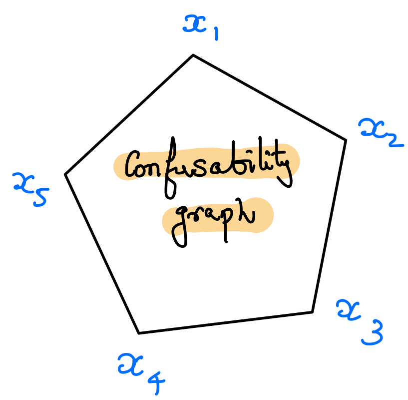
$\Theta(G)$ and Shannon capacity of G

Channel

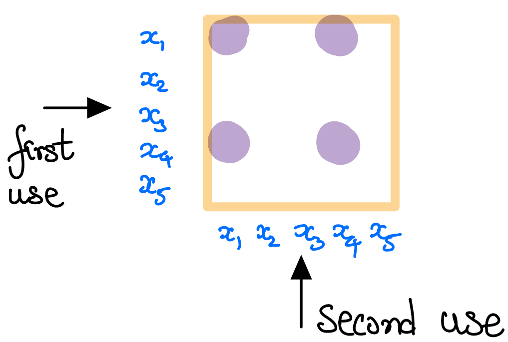


E.g. The 5/2 channel

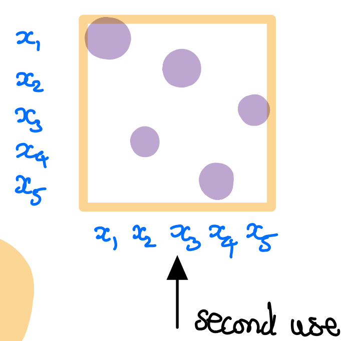




- With one use of the channel we can send two messages. One bit of information per use.
- With two uses of the channel

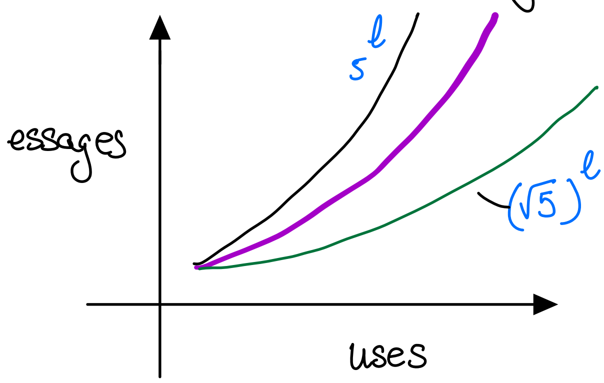


$\frac{1}{2}\log_2 5$ bits ≈ 1.161



- With 4 uses of the channel we can send 25 messages.
- With $2k$ uses, we can send $(5)^k$ messages.

$\boxed{\text{Can we do better?}}$



m ≃ the true rate of growth

$G \otimes H$

$$V(G \otimes H) = V(G) \times V(H)$$

$$E(G \otimes H) = \Big\{ \{(v_1,u_1),(v_2,u_2)\} : \{v_1,v_2\} \in E(G) \text{ or } v_1 = v_2 \text{ and } \{u_1,u_2\} \in E(H) \text{ or } u_1 = u_2 \Big\}$$

Claim:

messages with k uses $= \alpha(G^{\otimes k})$

$$\boxed{\text{Shannon Capacity: } \lim_{k\to\infty} \frac{1}{k}\, \alpha(G^{\otimes k})}$$

The limit exists, see the textbook.

Shown above

$$\text{Lovász} \Rightarrow \alpha(C_5^{\otimes k}) \le \theta(C_5^{\otimes k}) \le \theta(C_5)^k \le (\sqrt{5})^k.$$

general fact below

particular to C_5

__Claim:__ $\Theta(G \otimes H) \leq \Theta(G)\,\Theta(H)$

Suppose $(u_1, \ldots, u_m, c) \in \mathbb{R}^M$ achieve $\Theta(G)$

and $(\tilde{u}_1, \ldots, u_{\tilde{m}}, \tilde{c}) \in \mathbb{R}^{\tilde{M}}$ achieve $\Theta(H)$.

For $G \otimes H$, define $U_{ij} = u_i \tilde{u}_j^T$ and $C = c\tilde{c}^T$.

Treat these $m \times \tilde{m}$ matrices as vectors in $\mathbb{R}^{m\tilde{m}}$ and view the dot product as Hadamard product.

$$\boxed{\text{Recall: } M \circ \tilde{M} = \sum_{ij} m_{ij}\,\tilde{m}_{ij} = \operatorname{Tr} M^T \tilde{M}}$$

Then,
$$
\begin{aligned}
(u_i \tilde{u}_j^T) \circ (u_k \tilde{u}_\ell^T) &= \operatorname{Tr} (u_i \tilde{u}_j^T)^T u_k \tilde{u}_\ell^T \\
&= \operatorname{Tr} \tilde{u}_j \underbrace{u_i^T u_k}\, \tilde{u}_\ell^T \\
&= \operatorname{Tr} \langle u_i, u_k \rangle \tilde{u}_j \tilde{u}_\ell^T \\
&= \langle u_i, u_k \rangle \operatorname{Tr} \tilde{u}_j \tilde{u}_\ell^T \\
&= \langle u_i, u_k \rangle \langle \tilde{u}_j, \tilde{u}_\ell \rangle
\end{aligned}
$$

This calculation shows that

$$\left.\begin{array}{l} \bullet\ \{(i,j),(k,\ell)\} \notin E(G \otimes H) \Rightarrow U_{ij} \perp U_{k,\ell} \\ \bullet\ U_{ij} \circ C = \langle u_i, c \rangle \langle \tilde{u}_j, \tilde{c} \rangle \end{array}\right\} \Rightarrow \text{CLAIM}$$

CLAIM: $\theta(C_5) \leq \sqrt{5}$

Assume that the vertices of C_5 are $\{0,1,2,3,4\}$.

Let $u_j = \frac{1}{\sqrt{1+z^2}}\begin{pmatrix} \cos 2\pi j/5 \\ \sin 2\pi j/5 \\ z \end{pmatrix}$, $\quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Set z so that $u_0 \perp u_2$, that is, $z^2 = -\cos\frac{4\pi}{5} = \frac{\sqrt{5}+1}{4}$

Then, $\frac{1+z^2}{z^2} = \sqrt{5}$ (Check!)

$$\boxed{\begin{aligned} \text{CLAIMS} \Rightarrow \alpha(C_5^{\otimes k}) &\leq \theta(C_5^{\otimes k}) \\ &\leq \theta(C_5)^k \\ &\leq (\sqrt{5})^k. \end{aligned}}$$