Lecture 15 : Max-2SAT

Max-SAT: Given a CNF formula ϕ, show a true/false assignment to the variables in ϕ so that the number of satisfied clauses is maximized.

This is an NP-hard problem. Interestingly, $\underbrace{\text{Max-2SAT}}_{\text{Every clause has} \leq 2 \text{ literals.}}$ is also NP-hard though 2SAT is in P.

There is a $3/4$- approximation algorithm for Max-2SAT. If every clause has exactly two literals then it is easy to see that a $\underbrace{\text{random assignment}}_{\text{each } X_i \text{ is set to true w.p. } \frac{1}{2} \text{ and false w.p. } \frac{1}{2}}$ leads to a $\frac{3}{4}$-approx. algorithm.

Even when there are clauses with a single literal, we can show a $\frac{3}{4}$-approx. algorithm (think about it).

An improved approximation algorithm

There is an obvious quadratic program for Max-2SAT.

$$\min \sum_{\text{clause } C} \frac{(1-y_i)(1-y_j)}{4}$$

this corresponds to clause $C = X_i \vee X_j$ evaluating to false.

$$\text{s.t.} \quad y_i^2 = 1 \text{ for } i = 1, \ldots, n$$
$$y_i \in \mathbb{R} \text{ for } i = 1, \ldots, n.$$

Note that this is not a strict quadratic program as there are degree 1 terms here.

In order to convert this to a strict quadratic program, let us introduce another variable y_0 which is also constrained to be ± 1.

$$\text{The variable } X_i = \begin{cases} \text{true} & \text{if } y_i = y_0 \\ \text{false} & \text{if } y_i = -y_0 \end{cases}$$

If clause $C = x_i$ then $\underbrace{v(C)}_{\text{value of } C} = \dfrac{1 + y_0 y_i}{2}$

and if $C = \overline{x_i}$ then $v(C) = \dfrac{1 - y_0 y_i}{2}$.

Suppose $C = x_i \vee x_j$. Then
$$v(C) = 1 - \frac{(1-y_0y_i)}{2}\frac{(1-y_0y_j)}{2}$$
$$= \frac{1}{4}\left(3 + y_0y_i + y_0y_j - y_0^2 y_i y_j\right)$$
$$= \frac{1+y_0y_i}{4} + \frac{1+y_0y_j}{4} + \frac{1-y_iy_j}{4}.$$
(note that we used $y_0^2 = 1$ here)

It is easy to check in all cases that the value of a 2 literal clause consists of a linear combination of terms of the form $(1 + y_iy_j)$ or $(1 - y_iy_j)$.

Therefore, a Max-2SAT instance can be written as the following strict quadratic program, where the a_{ij}'s and b_{ij}'s are appropriate coefficients.

$$\text{maximize} \quad \sum_{i,j} a_{ij}(1 + y_iy_j) + b_{ij}(1 - y_iy_j)$$
$$\text{s.t.} \quad y_i^2 = 1 \quad \text{for } i = 0, 1, \dots, n.$$
$$y_i \in \mathbb{R} \quad \text{for } i = 0, 1, \dots, n.$$

Vector program relaxation: We go from $\mathbb{R}$ to $\mathbb{R}^{n+1}$. So the variable $y_i \in S^0 \rightsquigarrow$ vector $\vec{u_i} \in S^n$.

$$\max. \sum_{i,j} a_{ij}(1 + (u_i, u_j)) + b_{ij}(1 - (u_i, u_j))$$
$$\text{s.t.} \quad (u_i, u_i) = 1 \quad \text{for } i = 0, 1, \dots, n.$$
$$u_i \in \mathbb{R}^{n+1} \quad \text{for } i = 0, 1, \dots, n.$$

For convenience, we will write u_i instead of $\vec{u_i}$

Similar to max-cut, we solve the above vector program by solving the equivalent SDP. Let $u_0^*, u_1^*, \ldots, u_n^*$ be the optimal solution of the vector program.

Randomized rounding

Pick a point $\vec{r}$ uniformly at random on the unit sphere S^n in $(n+1)$ dimensions.

Let $y_i = 1$ if $(r, u_i^*) = 1$ and $y_i = -1$ otherwise.

Let C be the random variable denoting the number of clauses satisfied by this truth assignment.

Claim. $E[C] \geq (0.87856) \cdot OPT_v$, where OPT_v is the optimum value of the SDP.

Proof. $E[C]$

$$= 2 \sum_{i,j} a_{ij} \Pr[y_i = y_j] + 2 \sum_{i,j} b_{ij} \Pr[y_i \neq y_j].$$

Let θ_{ij} denote the angle between u_i^* and u_j^*.

$$\Pr[y_i \neq y_j] = \frac{\theta_{ij}}{\pi} \geq \frac{\alpha}{2}(1 - \cos\theta_{ij})$$ (by our earlier analysis)

where $\alpha = 0.87856$.

$$\Pr[y_i = y_j] = \frac{2\pi - 2\theta_{ij}}{2\pi} = \frac{\pi - \theta_{ij}}{\pi} \geq \frac{\alpha}{2}(1 + \cos\theta_{ij}).$$

The claim $\frac{\pi - \theta_{ij}}{\pi} \geq \frac{\alpha}{2}(1 + \cos\theta_{ij})$ follows from

$$\frac{2}{\pi} \cdot \frac{(\pi - \theta_{ij})}{(1 + \cos\theta_{ij})} = \frac{2}{\pi} \cdot \frac{\phi_{ij}}{(1 - \cos\phi_{ij})}$$ by substituting $\phi_{ij} = \pi - \theta_{ij}$.

We already know that $\frac{2}{\pi} \frac{\phi_{ij}}{(1 - \cos\phi_{ij})} \geq \alpha$.

Thus $E[C] \geq \alpha \cdot \sum_{i,j} \left(a_{ij}(1+\cos\theta_{ij}) + b_{ij}(1-\cos\theta_{ij}) \right)$

$$= \alpha \cdot OPT_v$$

As done for max-cut, we can repeat this algorithm an appropriate number of times and return the best assignment.

Thus w.p. $\geq 3/4$, we can return an assignment that satisfies $\geq 0.878 \cdot opt$ many clauses, where opt is the maximum number of clauses that can be simultaneously satisfied.

Conic Programming and Duality

A linear program in equational form is an optimization problem of the form:

$$\min \ c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0.$$

In conic programming we replace $x \geq 0$ with $x \in K$ for some closed convex cone K.

A conic program in equational form is written as:

$$\inf \ c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$x \in K.$$

Let V be a real and finite dimensional vector space.

Let $K \subseteq V$ be a non-empty closed set. This means the complement of K is an open set. That is, for every point x in $V \setminus K$,

Let us now prove that: $K \subseteq SYM_n$ of positive semidefinite matrices is a closed convex cone.
(Please check this for the first two examples.)

Proof. If $x^T M x \geq 0$ then $x^T \lambda M x = \lambda x^T M x \geq 0$ for $\lambda \geq 0$.

Also, if $x^T M x \geq 0$ and $x^T N x \geq 0$ then $x^T (M + N) x \geq 0$.

To show closedness, we check that the complement is open. Let M be a symmetric matrix that is not positive semidefinite. Then $\exists x \in \mathbb{R}^n$ s.t. $x^T M x < 0$ and this inequality holds for all matrices M' in a sufficiently small neighborhood of M.

4. The toppled ice cream cone in $\mathbb{R}^3$

$$K = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ xy \geq z^2\}.$$

Claim. K is a closed convex cone.

The above claim follows from the observation that K can alternatively be defined as the set of all (x, y, z) s.t. the symmetric matrix $\begin{pmatrix} x & z \\ z & y \end{pmatrix}$ is positive semidefinite.

Observe that in conic programs, we now use infimum instead of minimum. Consider the following example.

$$\begin{aligned} \inf\ & x_2 + x_3 \\ \text{s.t. } & x_1 = 1 \\ & x \in K_{ice} \end{aligned}$$ (the ice cream cone in $\mathbb{R}^3$).

So we have

$$\left.\begin{aligned} & x_3^2 \geq x_1^2 + x_2^2 \\ & \text{and } x_3 \geq 0 \end{aligned}\right\} \iff x_1^2 \leq (x_3 - x_2)\cdot(x_3 + x_2) \iff 1 \leq (x_3 - x_2)(x_3 + x_2).$$

For any $\varepsilon > 0$, set $x_3 = \frac{1}{2}\left(\varepsilon + \frac{1}{\varepsilon}\right)$ and $x_2 = \frac{1}{2}\left(\varepsilon - \frac{1}{\varepsilon}\right)$.
The objective fn. can be made arbitrarily small but cannot be 0.

there exists a positive real number ε (depending on x) s.t. the open ball $B(x, \varepsilon)$ centered at x of radius ε is contained in $V \setminus K$.

K is called a closed convex cone if the following two conditions hold.

(i) If $x \in K$ then $\lambda x \in K$ for all non-negative real numbers λ.

(ii) If $x, y \in K$ then $x + y \in K$.

Some examples of closed convex cones

1. The non-negative orthant $K = \{x \in \mathbb{R}^n : x \geqslant 0\}$.

2. The ice cream cone (also called the Lorentz cone or the second-order cone)

$$K = \{x \in \mathbb{R}^n : x_n^2 \geqslant \sum_{i=1}^{n-1} x_i^2, \quad x_n \geqslant 0\}.$$

3. The positive semidefinite cone

$$K = \{X \in \underbrace{SYM_n}_{\substack{\text{the set of} \\ n \times n \text{ real symmetric matrices.}}} : v^T X v \geqslant 0 \ \forall v \in \mathbb{R}^n\}.$$

In this case the conic program becomes

$$\inf \sum_{i,j} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{i,j} a_{ijk} x_{ij} = b_k \text{ for } k = 1, \ldots, m$$

$$X = (x_{ij}) \in K.$$

Duality (now our primal LP is a minimization problem)

For LP duality, we find a solution (y, s) such that $A^T y + s = c$ with $s \geqslant 0$. This implies:

$$c^T x = (A^T y + s)^T x = y^T A x + s^T x$$
$$= \underbrace{y^T b + s^T x \geqslant y^T b}.$$

This is because the primal and dual solutions require $x, s \geqslant 0$.

Thus we get the weak duality theorem. We would like to imitate the same argument for the dual of a conic program and show weak duality for conic programs. For this, we need $s \in K^*$ where

$$K^* = \{ s \in \mathbb{R}^n : s^T x \geqslant 0 \ \forall x \in K \}.$$

(Then we'll be able to show that $c^T x \geqslant y^T b$.)

K^* is the dual of the cone K. More formally, let $K \subseteq V$ be a closed convex cone. The set

$$K^* = \{ y \in V : (y, x) \geqslant 0 \ \forall x \in K \}$$

is called the <u>dual cone</u> of K.

Please show that K^* is again a closed convex cone.

<u>Some examples of dual cones</u>

1. Let $K = \{ x \in \mathbb{R}^n : x \geqslant 0 \}$. This is the non-negative orthant. Observe that $K^* = K$, i.e. the non-negative orthant is <u>self-dual</u>. It is easy to check that $K \subseteq K^*$. Let $y \notin K$. Then $y_i < 0$ for some $i \in \{1, .., n\}$. We have $y^T e_i < 0$ where e_i is the i-th unit vector. Since $e_i \in K$, $y \notin K^*$.

2. Let $K = \{x \in \mathbb{R}^n : x_n^2 \geq \sum_{i=1}^{n-1} x_i^2, \ x_n \geq 0\}$.

This is the ice cream cone in n dimensions.

This cone is also self-dual, i.e., $K^* = K$.

Let us first show that $K \subseteq K^*$. Let $x, y \in K$.

$$(y, x) = \sum_{i=1}^{n} y_i x_i = y_n x_n + \sum_{i=1}^{n-1} y_i x_i$$

$$\geq y_n x_n - \sqrt{\sum_{i=1}^{n-1} y_i^2} \cdot \sqrt{\sum_{i=1}^{n-1} x_i^2}$$

(by Cauchy-Schwartz inequality)

$$\geq 0 \quad \left(\text{since } x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2} \text{ and } y_n \geq \sqrt{\sum_{i=1}^{n-1} y_i^2}\right).$$

Thus $(y, x) \geq 0$ for all $x, y \in K$.

We now need to show that $K^* \subseteq K$.

Suppose $(y, x) \geq 0$ for all $x \in K$. Let us take $x_i = -y_i$ for $1 \leq i \leq n-1$ and $x_n = \sqrt{\sum_{i=1}^{n-1} y_i^2}$.

Then $(y, x) = y_n \cdot \sqrt{\sum_{i=1}^{n-1} y_i^2} - \sum_{i=1}^{n-1} y_i^2 \geq 0$

$$\Rightarrow \text{ either } \underbrace{\sum_{i=1}^{n-1} y_i^2 = 0}_{} \text{ or } \underbrace{y_n \geq \sqrt{\sum_{i=1}^{n-1} y_i^2}}_{\text{thus } y \in K}.$$

To make this claim, take $x = (0, \dots, 0, 1)$ ← then $y_n \geq 0$, so $y \in K$.

In both cases we get $y \in K$. Thus $K^* \subseteq K$.

3. Let K be the positive semidefinite cone.

So $K = \{X \in SYM_n : v^T X v \geq 0 \ \forall v \in \mathbb{R}^n\}$.

Let us show this cone is also self-dual.

Let us first show that $K \subseteq K^*$. So for any two positive semidefinite matrices X and Y, we need to show $\underbrace{Tr(X^TY) = Tr(XY)}_{\text{recall that } X \text{ is symmetric}} \geq 0$.

A useful fact: If A is a positive semidefinite matrix then it has a unique positive semidefinite squareroot, i.e $A = B^2$ where B is a psd matrix.

So $X = A^2$ and $Y = B^2$, where A and B are the (unique) psd square roots of X and Y, respectively.

Hence $Tr(XY) = Tr(A^2B^2) = Tr(A.(AB).B)$

$$= \underbrace{Tr(A.((AB).B)) = Tr((AB).(BA))}_{\text{since } Tr(MN) = Tr(NM) \text{ for any } M,N}$$

But $AB = (BA)^T$, hence $Tr(XY) = Tr(C^TC)$

$= \underbrace{\text{sum of squares}}_{\geq 0}$, where $C = BA$.

Thus $K \subseteq K^*$.

Now we need to show that $K^* \subseteq K$.

Let Y be a symmetric matrix that is not in K.

We need to construct a matrix $X \in K$ s.t. $Tr(XY)$ is negative.

We know that there is an orthogonal matrix O s.t. OYO^T = a diagonal matrix $\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ where some d_i's are negative

Let Z be the diagonal matrix with 0's and 1's on its diagonal — it has 0's exactly on those diagonal entries where $d_i > 0$ and it has 1's on those diagonal entries where $d_i \leq 0$. Note that Z is a psd matrix.

Let $X = O^T Z O$. So X is positive semidefinite.
That is, $X \in K$.

$$\begin{aligned} Tr(XY) &= Tr(O.X.O^T.O.Y.O^T) \\ &= Tr\left(Z.\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}\right) < 0. \end{aligned}$$

(by construction)

Thus $Y \notin K^*$.

4. Let K be the toppled ice cream cone. Its dual cone

$$K^* = \left\{(x, y, z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ xy \geq \frac{z^2}{4}\right\}.$$

Observe that this is a vertically stretched version of K.

(Please do this as an exercise.)

__Lemma.__ Let $K \subseteq V$ be a closed convex cone.
Then $(K^*)^* = K$.

It is easy to see that $K \subseteq (K^*)^*$. Let $x \in K$.
By the definition of K^*, $(y, x) \geq 0$ for all $y \in K^*$.
Since $(x, y) = (y, x)$, it follows that $x \in (K^*)^*$.
For the other direction, we need a __separation theorem__. (next lecture)

__References__

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2. Approximation Algorithms.
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