Cone programs

Consider a linear program.

$$\text{maximize} \quad c^T x$$

$$\text{subject to} \quad b - Ax \geq 0$$

$$x \geq 0$$

There are two cones here:

$K = (\mathbb{R}^+)^n$

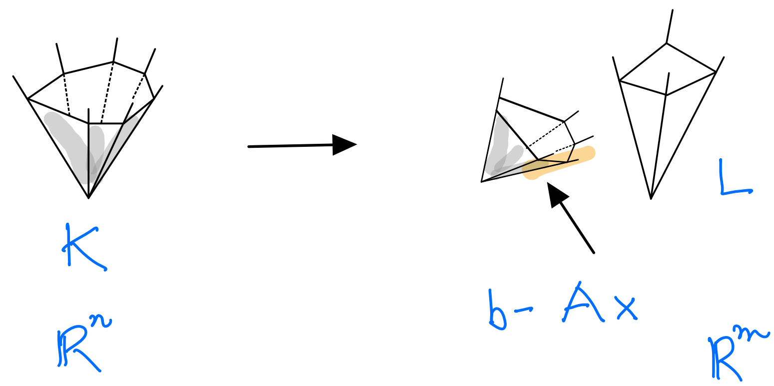
$L = (\mathbb{R}^+)^m$

$A \in \mathbb{R}^{m \times n}$

$c, x \in \mathbb{R}^n$

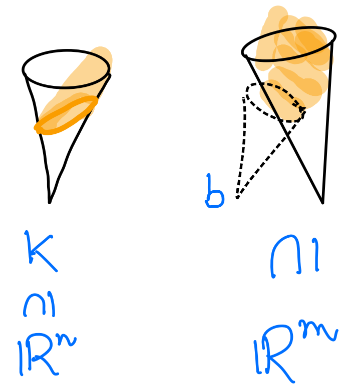
$b \in \mathbb{R}^m$

$Ax \leq b$



If we now overlook the fact that K and L were special cones $(\mathbb{R}^+)^n$ and $(\mathbb{R}^+)^m$, then we get general cone programs.

$$P: \quad \boxed{\begin{array}{ll} \text{maximize} & \langle c, x \rangle \\ \text{subject to} & b - A(x) \in L \\ & x \in K \end{array}}$$



K, L: closed, convex

- Value of a feasible cone program: $L = \{0\}$

$$\sup \left\{ \langle c, x \rangle : \begin{array}{l} b - A(x) \in L \\ x \in K \end{array} \right\}$$

- Optimal solution

  A feasible solution x^* such that

  $$\langle c, x^* \rangle \geq \langle c, x \rangle \text{ for all feasible } x.$$

> There are cone programs with finite value but no optimal solution.

Example

$L = \{0\}$

minimize x

$1 - A\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in L$

subject to $z = 1$

K = Toppled icecream cone

$\left.\begin{array}{l} z^2 \leq xy \\ x, y \geq 0 \end{array}\right\}$ Toppled icecream cone

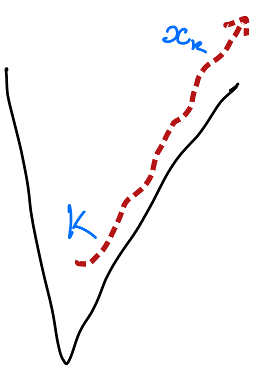
$A = +\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, $b = (1)$

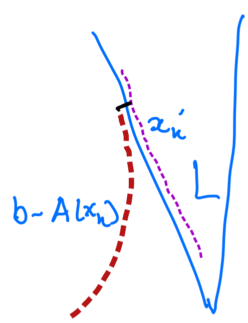
Value $= 0$, no optimal solution.

Definition: A sequence $(x_k)_{k \geq 1}$, $x_k \in K$, is feasible if there is a sequence $(x'_k)_{k \geq 1}$, $x'_k \in L$, such that

$$\lim_{k \to \infty} \left(b - A(x_k)\right) - x'_k = 0$$

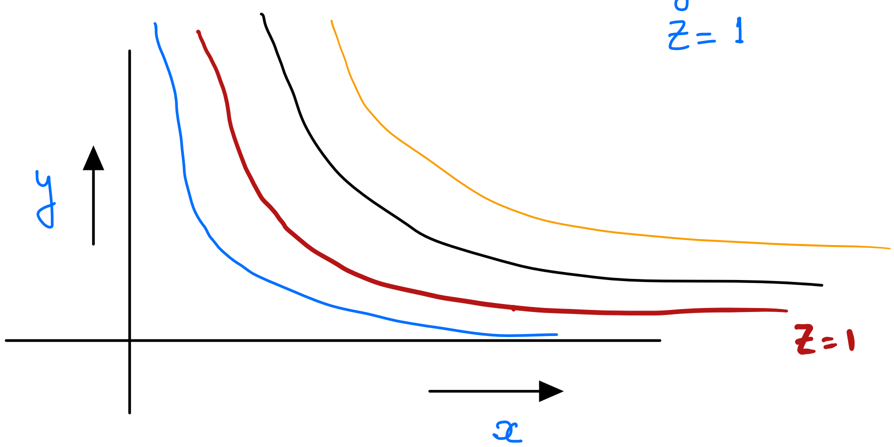
$x \in K$ feasible $\Rightarrow$ $(x, x, \dots)$ limit feasible.





Example: maximize x

$$\text{subject to} \quad \left.\begin{aligned} & z^2 \le xy \\ & x, y \ge 0 \\ & y = 0 \\ & z = 1 \end{aligned}\right\} \text{infeasible}$$



$$K = \{ z^2 \le xy : x, y \ge 0 \}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (x, y, z) \longmapsto (y, z)$$

$$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad p_k = \begin{pmatrix} k \\ 1/k \\ 1 \end{pmatrix}, \quad p_k' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$L = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} - A(p_k) = \begin{pmatrix} -1/k \\ 0 \end{pmatrix} \xrightarrow[k \to \infty]{} 0$$

So it is limit feasible.

Limit value

$$\sup_{\substack{(x_k)_{k\geq 1} \\ \text{limit feasible}}} \lim_{k\to\infty} \langle c, x_k \rangle$$

$$\boxed{\begin{array}{ll} \text{maximize} & z \\ \text{subject to} & (x, 0, z) \in \text{toppled icecream cone} \end{array}}$$

Value $= 0$

Limit value $= \infty$

$K = \mathbb{R}_+^3$

$L =$ toppled icecream cone

$$A = \begin{pmatrix} -1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$b - Ax = 0$$

$$b - \left(A \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$$

$$p_k = \begin{pmatrix} k^3 \\ 1/k \\ k \end{pmatrix}, \quad p_k' = \begin{pmatrix} k^3 \\ 1/k \\ k \end{pmatrix}$$

The book has a different formulation.

$K = \mathbb{R}^2$

$b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$L =$ toppled icecream cone

$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$

$c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Take $\quad p_k = \begin{pmatrix} k^3 \\ k \end{pmatrix} \quad p_k' = \begin{pmatrix} k^3 \\ 1/k \\ k \end{pmatrix}$

Check!

$\boxed{\text{Luckily, interior points force the value and limit value of cone programs to match.}}$

Interior points

An interior point (or Slater point) of the cone program

$$\boxed{\begin{array}{ll} \text{maximize} & \langle c, x\rangle \\ \text{subject to} & b - A(x) \in L \\ & x \in K \end{array}}$$

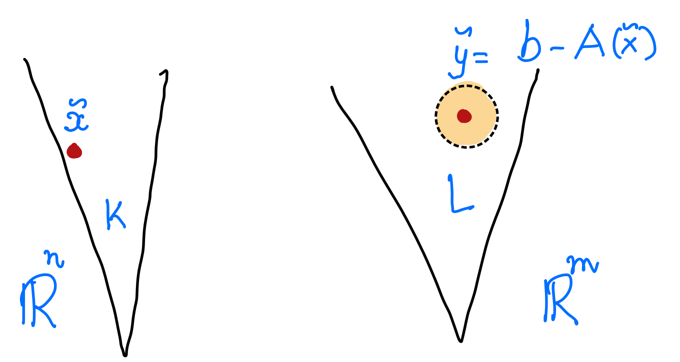
is a point x such that

$$x \in K \quad \text{and} \quad b - A(x) \in L$$

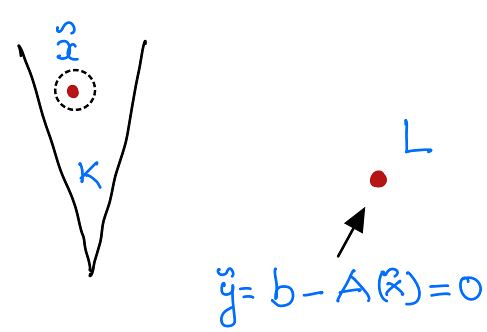
and

$$\begin{array}{ll} x \in \operatorname{int}(K) & \text{if } L = \{0\} \\ b - A(x) \in \operatorname{int}(L) & \text{otherwise.} \end{array}$$

L is full-dimensional, $L \neq \{0\}$



$L = \{0\}$



Theorem: If the cone program P

$$\boxed{\begin{array}{ll} \text{maximize} & \langle c, x\rangle \\ \text{subject to} & b - A(x) \in L \\ & x \in K \end{array}}$$

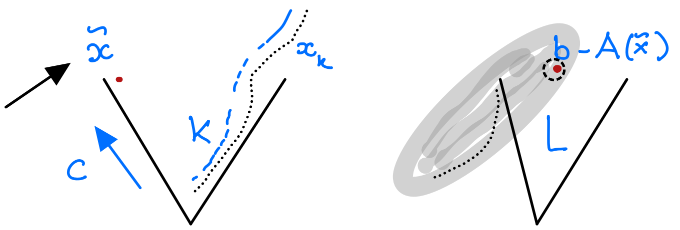
$\mathrm{Value}(P) \geq \mathrm{limitvalue}(P) - \varepsilon \quad \forall \varepsilon > 0$

has an interior point, then $\mathrm{value}(P) = \mathrm{limit\ value}(P)$.

Proof: Suppose $(x_k)_{k \geq 0}$ attains the limit value.

Case $L \neq \{0\}$:

the interior point



Idea: Shift x_k towards $\mathring{x}$ to obtain a feasible point for the cone program.

$$\omega_k = \eta\, \mathring{x} + (1-\eta)\, x_k$$

- For all $\eta > 0$, the points ω_k eventually become feasible, i.e., for $k \geq k_0$ we have ω_k is feasible.
- Given $\varepsilon > 0$, may choose η small enough so that

$$\langle c, x_k\rangle \geq (1-\varepsilon)\langle c, w_j\rangle - \varepsilon \qquad \textbf{(Homework)}$$

Then, $\forall \varepsilon > 0$

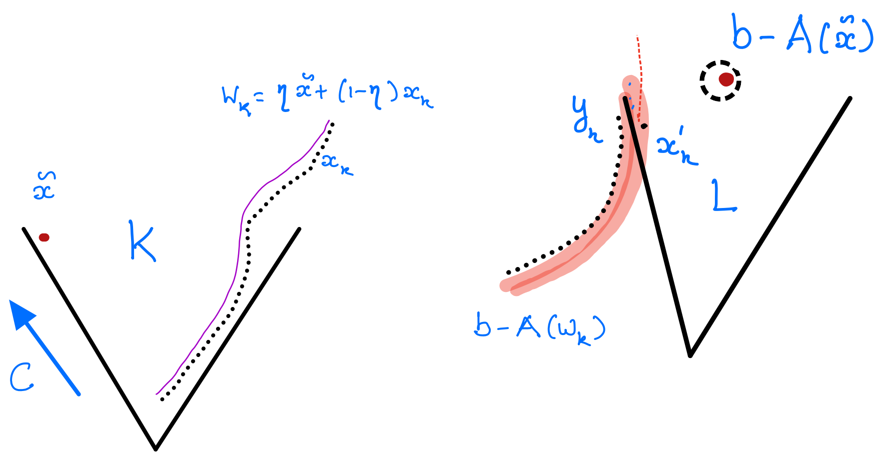
$$\langle c, w_k \rangle \geq (1-\varepsilon)\langle c, w_k \rangle - \varepsilon$$

$$\Downarrow$$

$$\text{value}(P) \geq \limsup_{k\to\infty} \langle c, w_k \rangle \geq \limsup (1-\varepsilon)\langle c, x_k \rangle - \varepsilon$$

$$\Downarrow$$

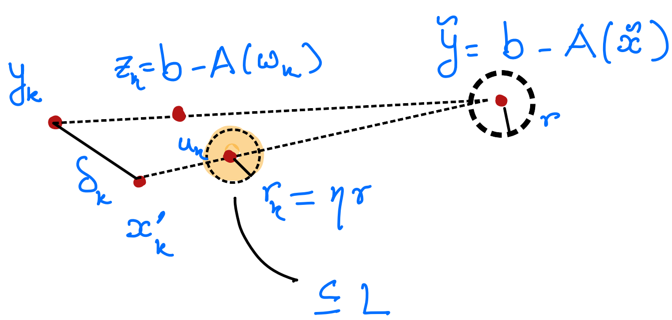
$$\text{value}(P) = \text{limit value}(P)$$



Why must ω_k eventually become feasible?

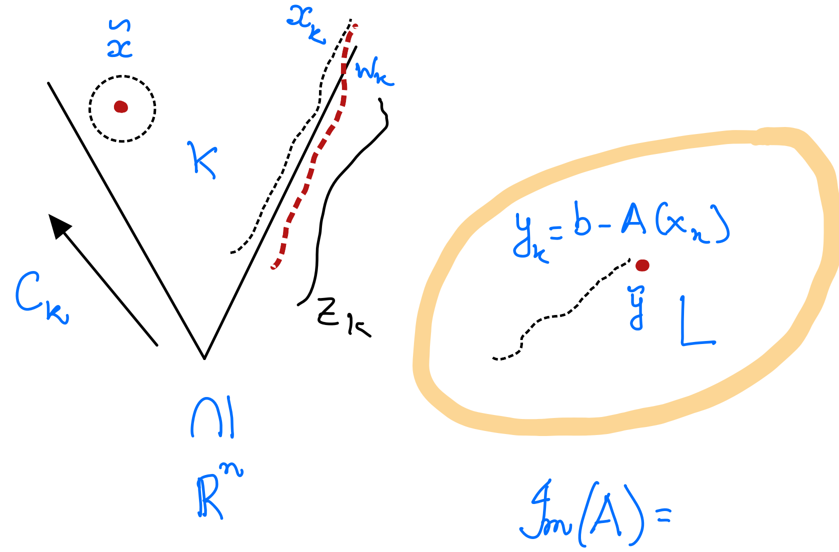
- $\text{dist}(u_k, z_k) = (1-\eta)\,\delta_k$
- eventually

$(1-\eta)\,\delta_k \leq r_k = \eta r$



So, $z_k \in L$. $\Rightarrow$ ω_k is feasible.

Case $L = \{0\}$:



Suppose $A(v_1) = u_1$

$A(v_2) = u_2$

$\vdots$

$A(v_m) = u_m$

$\operatorname{Im}(A) = V'' = \operatorname{span}\{u_1, u_2, \dots, u_m\}$

linearly independent

$V' = \operatorname{span}\{v_1, v_2, \dots, v_m\} \subseteq \mathbb{R}^n$

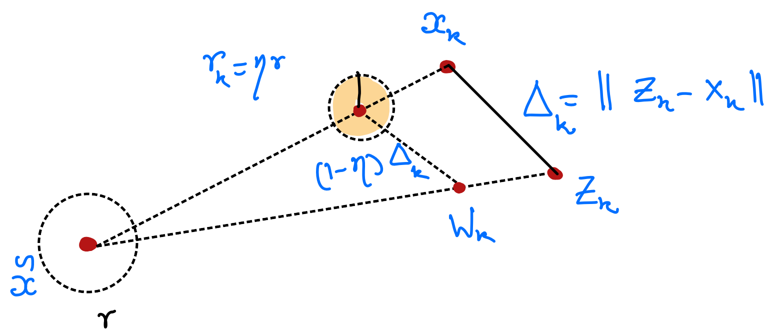
$A : V' \to V''$ is a bijection.

Consider $z_k = x_k + A^{-1}(b - A(x_k)) = x_k + A^{-1}(A(\tilde{x}) - A(x_k))$

$w_k = \eta \tilde{x} + (1-\eta) z_k$

- w_k is feasible eventually, i.e., for all $k \geq k_0$.
- $\langle c, w_k \rangle \geq (1-\varepsilon)\langle c, x_k \rangle - \varepsilon$, for all $k \geq k_0$.

if η is chosen appropriately (**Homework**)



Eventually, $(1-\eta)\,\Delta_k \leq r_k \rightarrow w_k \in K$

$$|\langle c, x_k - w_k \rangle| \leq |\langle c, \eta(\tilde{x} - x_k)\rangle| + |\langle c, (1-\eta)\Delta_k|$$

What does this lead to?

DUALS

maximize $\langle c, x\rangle$	minimize $\langle b, y\rangle$
subject to $b - A(x) \in L$	subject to $A^T(y) - c \in K^*$
$x \in K$	$y \in L^*$

Strong duality theorem of cone programming

If the primal is feasible, has finite value γ, and has an interior point $\tilde{x}$, then the dual is also feasible and has the same value γ.