Linear programs

maximize $c^T x$
subject to $Ax \leq b$

Equational form

maximize $c^T x$
subject to $Ax = b$ ⟵ equalities
$x \geq 0$ ⟵ the only inequalities

Transforming an arbitrary linear program to equational form

- Inequalities to equalities

slack variable

$$2x_1 - x_2 \leq 4 \rightsquigarrow 2x_1 - x_2 + z_1 = 4$$
$$z_1 \geq 0$$

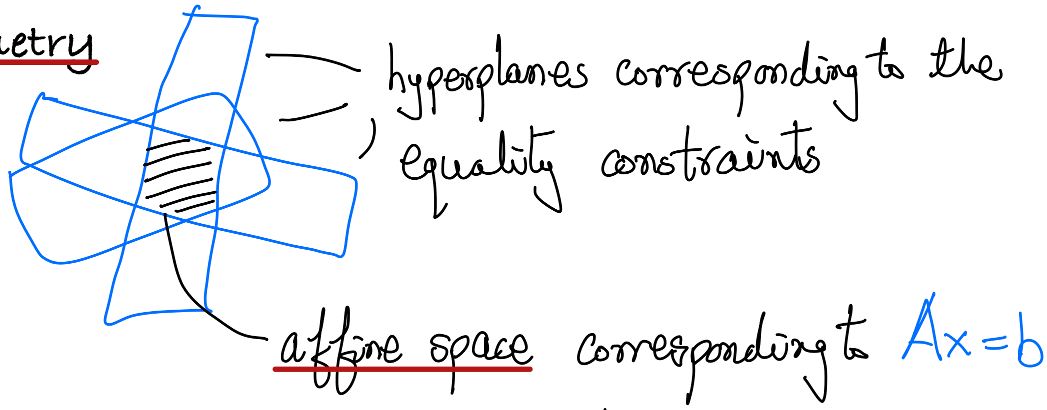
$$x_1 + 3x_2 \geq 4 \rightsquigarrow x_1 + 3x_2 - z_2 = 4$$
$$z_2 \geq 0$$

- Uconstained variables

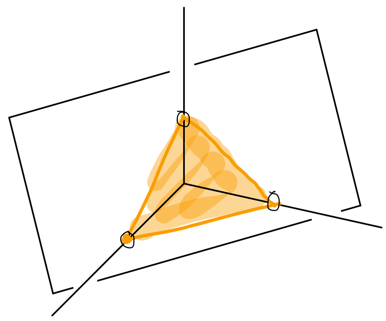
For each variable x_i introduce two new variables y_i and z_i. Replace x_i by $y_i - z_i$ everywhere, and add the non-negativity constraints

$$y_i \geq 0 \quad \text{and} \quad z_i \geq 0$$

Geometry



The feasible region is the intersection of this affine space with the positive orthant corresponding to $x \geq 0$

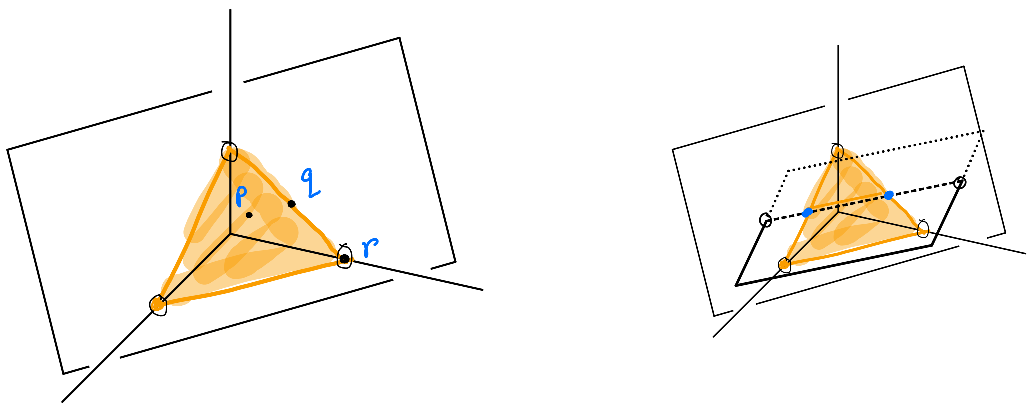


Assumptions

- The system $Ax=b$ has at least one solution;
- The rows of the matrix A are linearly independent

> A is an $m \times n$ matrix with m linearly independent rows.

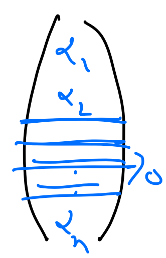
Basic feasible solutions



For an $m \times n$ matrix A and a subset B of $[n] = \{1, 2, \dots, n\}$ let A_B denote the matrix consisting of those columns of A whose indices appear in B.

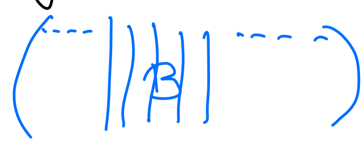
A basic feasible solution of the linear program

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \qquad (A \in \mathbb{R}^{m \times n}) \\ & x \geq 0 \end{aligned}$$



is a feasible solution $x \in \mathbb{R}^n$ for which there exists an m-element set B such that

- columns of A_B are linearly independent;
- $x_j = 0$ for $j \notin B$.

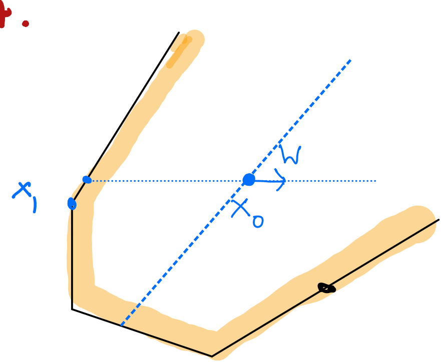


Proposition: For every $B \subseteq \{1,2,\dots,n\}$ with A_B non-singular, there exists at most one feasible solution $x \in \mathbb{R}^n$ with $x_j = 0$ for all $j \notin B$. $Bx_B = b$

Theorem: If the objective function of an LP is bounded above, then for every feasible solution x_0, there is a basic feasible solution $\tilde{x}$, s.t.

Presented again in Lecture 4.

$$c^T \tilde{x} \geq c^T x_0 .$$



$$Ax_0 = b$$

$$K = \{ j : x_0[j] > 0 \}$$

If A_K is singular, there is a $w \neq 0$ supported on K s.t. $Aw = 0$.

Corollary:

LP feasible and bounded

$\Downarrow$

$\exists$ optimum solution

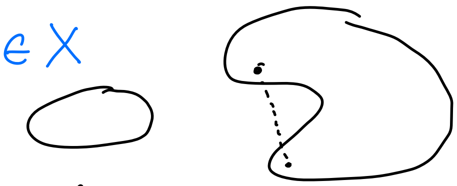
$\Downarrow$

$\exists$ basic feasible optimum solution

Convexity and convex polyhedra

$X \subseteq \mathbb{R}^n$ is convex if $\forall\, u, v \in X$ $\forall\, t \in [0,1]$

$$tu + (1-t)v \in X$$



$S \subseteq \mathbb{R}^n$: The corex hull of S is the smallest convex set containing S, that is, the intersection of all convex sets that contain S.

A convex combination of points $x_1, x_2, \dots, x_k \in \mathbb{R}^n$ is a point x that can be expressed as

$x = t_1 x_1 + t_2 x_2 + \dots + t_k x_k$ where $t_i \geq 0$
and $t_1 + t_2 + \dots + t_k = 1$.

Proposition: Let $X \subset \mathbb{R}^n$. Then,

Convex hull of X = $\{ x \in \mathbb{R}^n :$ x is a convex combination of some finite set of points in $X \}$.

hyperplane = $\{x \in \mathbb{R}^n : \; a^T x = b\}$, $\underset{\mathbb{R}^n}{\underset{\cap}{a}} \neq 0$, $\underset{\mathbb{R}}{\underset{\cap}{b}}$

closed half-space = $\{x \in \mathbb{R}^n : \; a^T x \geq b\}$

A convex polyhedron is an intersection of finitely many closed half-spaces.

A bounded convex polyhedron is a convex polytope.

The dimension of a convex polyhedron $P \subseteq \mathbb{R}^n$ is the smallest dimension of an affine subspace containing P.

Examples

- Cubes
- Crosspolytope
- Simplex

Vertices and basic feasible solutions

P: a polyhedron

A point v in P is a vertex of P if there is a nonzero vector c s.t.

$$c^T v > c^T y \quad \forall y \in P \setminus \{v\}.$$

The $\mathcal{H} = \{x \in \mathbb{R}^n : c^T x = c^T v\}$ intersects P exactly at v, an P lies entirely in one of the closed half-spaces defined by $\mathcal{H}$.

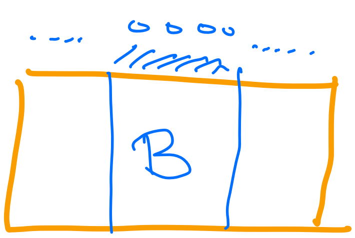
Theorem: P: the set of all feasible solutions of a linear program in equational form.

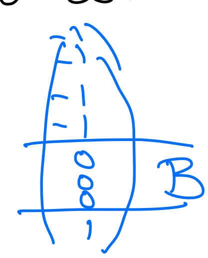
Convex polyhedron

(i) v is a vertex of P

$$\Updownarrow$$

(ii) v is a basic feasible solution of the LP.



$c =$ 

Arbitrary linear programs

A basic feasible solution of a linear program

$$\text{maximize} \quad c^T x$$

$$\text{subject to} \quad Ax \ \substack{\leq \\ =} \ b$$

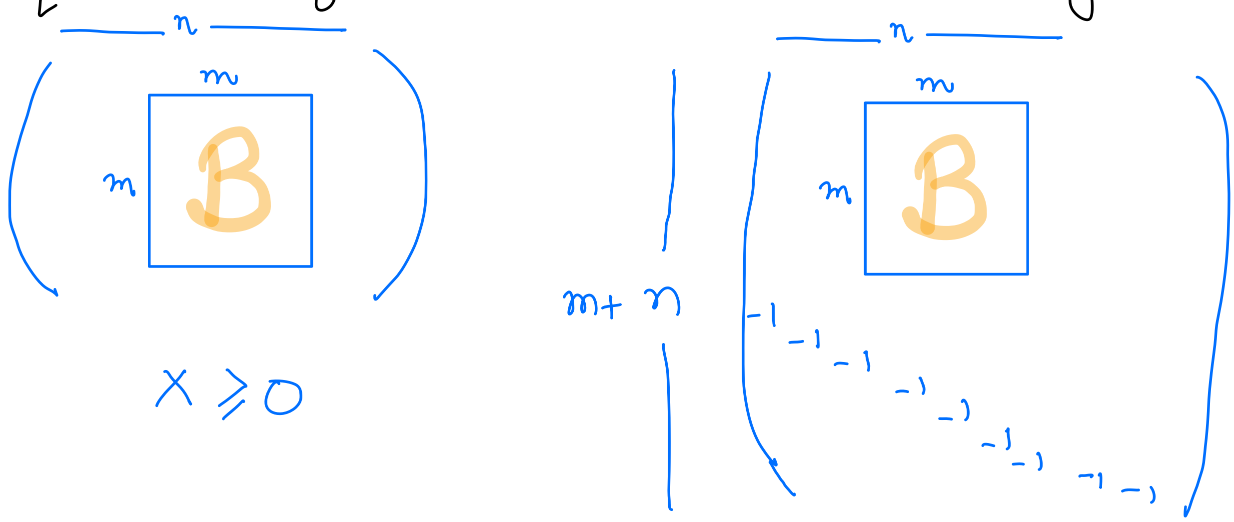
$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$
$c \in \mathbb{R}^n$

(not necessarily in equational form)

is a basic feasible solution for which some n linearly independent constraints are satisfied with equality.

equational form

arbitrary



$x \geq 0$

- There are (general) LPs none of whose infinitely many optimal solutions is basic.

- Vertices and extremal points

vertex – A point in the convex set where some linear function attains its unique maximum.

extreme point – A point in the convex set that cannot be written as a convex combination of two other points in the convex set.

For convex polyhedra x is a vertex

$$\Updownarrow$$

x is an extreme point

The main theorem of convex polytopes

A polytope is equal to the convex hull of its set of vertices.

Not obvious!