Last time

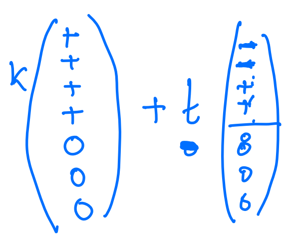
- LPs in equational form
- Basic feasible solutions
- Convex polyhedra, vertices

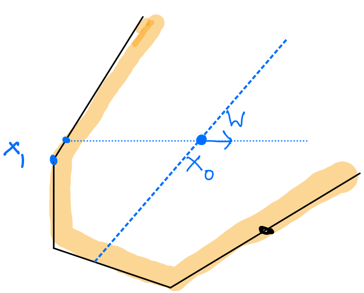
$$Ax = b$$

Theorem: If the objective function of an LP is bounded above, then for every feasible solution x_0, there is a basic feasible solution $\tilde{x}$, s.t.

$$c^T \tilde{x} \geq c^T x_0 .$$

x_0





$m \times n$

$$A x_0 = b$$

$$K = \{ j : x_0[j] > 0 \}$$

(1) If A_K is non-singular, we are done.

(2) If A_K is singular, there is a $w \neq 0$ supported on K s.t. $Aw = 0$.

(2a) $c^T w = 0$

We move either along w or along $-w$ until we reach an x_1 with more zeros. Repeat.

(2b) $c^T w \neq 0$, by replacing w by $-w$ if necessary, assume $c^T w < 0$. By moving along $-w$, we reach x_1 as before.

Today

Solving LPs using the simplex method

maximize $x_1 + x_2$

Subject

$$
\begin{aligned}
-x_1 + x_2 + x_3 \quad\quad\quad &= 1 \\
x_1 \quad\quad\quad + x_4 \quad &= 3 \\
x_2 \quad\quad\quad + x_5 &= 2 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

$$Ax = b$$

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad B$$

A basis allows one to declare some variables as dependent variables

$$b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

1

$$
\begin{aligned}
x_3 &= 1 + x_1 - x_2 \\
x_4 &= 3 - x_1 \\
x_5 &= 2 \quad\quad - x_2 \\
\hline
z &= x_1 + x_2
\end{aligned}
$$

2

$$
\begin{aligned}
x_3 &= 4 - x_2 - x_4 \\
x_1 &= 3 \quad\quad - x_4 \\
x_5 &= 2 - x_2 \\
\hline
z &= 3 + x_2 - x_4
\end{aligned}
$$

3

$$
\begin{aligned}
x_3 &= 2 - x_4 + x_5 \\
x_1 &= 3 - x_4 \\
x_2 &= 2 - x_5 \\
\hline
z &= 5 - x_4 - x_5
\end{aligned}
$$

When simplex gets stuck

(1) Unbounded LPs

$$\max \quad x_1$$

$$\text{subject to} \quad \begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

1

$$\begin{aligned} x_3 &= 1 - x_1 + x_2 \\ x_4 &= 2 + x_1 - x_2 \\ \hline z &= \phantom{1 +{}} x_1 \end{aligned}$$

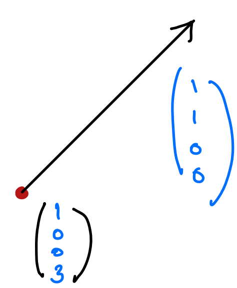
2

$$\begin{aligned} x_1 &= 1 + x_2 - x_3 \\ x_4 &= 3 - x_3 \\ \hline z &= 1 + x_2 - x_3 \end{aligned}$$

??

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1+x_2 \\ x_2 \\ 0 \\ 3 \end{pmatrix}} \quad \text{and} \quad z = 1 + x_2$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$



(2) Degeneracy

$$\begin{array}{l} x_3 = x_1 - x_2 \\ x_4 = 2 - x_1 \\ \hline z = x_2 \end{array} \longrightarrow \begin{array}{l} x_2 = x_1 - x_3 \\ x_4 = 2 - x_1 \\ \hline z = x_1 - x_3 \end{array}$$

Cycling

$$\begin{array}{l} x_2 = 2 - x_3 - x_4 \\ x_1 = 2 - x_4 \\ \hline z = 2 - x_3 - x_4 \end{array}$$

(3) Where to start?

maximize $x_1 + 2x_2$

subject to
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_2 + x_3 &= 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Assume RHS ≥ 0.

- To the i^{th} equation add a new variable y_i.
- Set $y_i = b_i$.
- Solve an auxiliarly LP to

minimize $y_1 + y_2 + \dots + y_m$

i.e. maximize $-y_1 - y_2 - \dots - y_m$

Original LP

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 3x_2 + x_3 = 4 \\ & 2x_2 + x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

auxiliary LP

$$\begin{aligned} \text{maximize} \quad & -y_1 - y_2 \\ \text{subject to} \quad & x_1 + 3x_2 + x_3 + y_1 = 4 \\ & 2x_2 + x_3 + y_2 = 2 \\ & x_1, x_2, x_3, y_1, y_2 \geq 0 \end{aligned}$$

- It is bounded because $-y_1 - y_2 \leq 0$.
- It has a feasible solution $y_1 = 4,\ y_2 = 2$, $x_1, x_2, x_3 = 0$
- So simplex method will find an optimum!
- If the optimum is zero, the values of the remaining variables is a basic feasible solution of the original LP.
- If not, the original LP is infeasible.

Solving the auxiliary LP

$$\begin{aligned} y_1 &= 4 - x_1 - 3x_2 - x_3 \\ y_2 &= 2 - 2x_2 - x_3 \\ \hline z &= -6 + x_1 + 5x_2 + 2x_3 \end{aligned}$$

$\downarrow$

$$\begin{aligned} y_1 &= 1 - x_1 - x_3 - \tfrac{3}{2}y_2 \\ x_2 &= 1 - \tfrac{1}{2}x_3 - \tfrac{1}{2}y_2 \\ \hline z &= -1 + x_1 - 0.5x_3 - 0.5y_2 \end{aligned}$$

$\downarrow$

$$\begin{aligned} x_1 &= 1 - x_3 - y_1 - \tfrac{3}{2}y_2 \\ x_2 &= 1 - \tfrac{1}{2}x_3 - \tfrac{1}{2}y_2 \\ \hline z &= 0 - 1.5x_3 - y_1 - 2y_2 \end{aligned}$$

So $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is a basic feasible solution for the original LP.

Back to the original LP

maximize $x_1 + 2x_2$

subject to
$$x_1 + 3x_2 + x_3 = 4$$
$$2x_2 + x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

$$x_1 = 1 + \frac{1}{2}x_3$$
$$x_2 = 1 - \frac{1}{2}x_3$$
$$z = 3 - \frac{1}{2}x_3$$

The feasible solution returned by the auxiliary LP was already optimal for the original LP.