The simplex tableau

The LP

maximize $c^T x$

subject to $Ax = b$

$x \geq 0$

$$\begin{array}{cc} & \overset{n}{\longleftrightarrow} \\ m & \left(\begin{array}{c:c} \overset{m\times m}{B} & N \end{array}\right) \end{array} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} b \end{pmatrix}$$

The tableau

$$\begin{array}{c} x_B = p + Q x_N \\ \hline z = z_0 + r^T x_N \end{array}$$

m basic variables

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$$

$n-m$ non-basic variables

$$p = A_B^{-1} b$$

$$Q = -A_B^{-1} A_N$$

$$z_0 = c_B^T p$$

$$r^T = c_N^T + c_B^T Q x_N$$

The simplex method

maximize $c^T x$

subject to $Ax = b, \; x \geq 0$

① To find the initial basic feasible solution, solve the LP

maximize $-y_1 - y_2 - \cdots - y_m$

subject to $Ax + I_m y = b; \; x, y \geq 0$

Infeasible

$\downarrow$ Feasible

② Start with the tableau

$$\begin{array}{rcl} x_B & = & p + Q x_N \\ \hline z & = & z_0 + r^T x_N \end{array}$$

$\longrightarrow$ $r \leq 0$ Optimal stop!

$\downarrow$

③ Select an entering variable: x_v s.t. $r_v > 0$.

Use a pivot rule to break ties.

④ Consider the rows of the tableau, where x_v has a negative coefficient.

If there is no such variable, the LP is unbounded.

Among them, pick the row j for which $-p_j / \text{coeff}(x_r)$ is minimum. The variable on the LHS will leave the basis.

Use a pivot rule to break ties.

⑤ Update the basis.

Pivot rules

1. Largest coefficient (Dantzig)

   Choose the incoming variable with the largest (positive) coefficient of the row of the objective function.

2. Largest increase

   Choose an incoming variable that leads to the largest improvement in the value of the objective function.

3. Steepest edge (a good rule in practice)

   Choose an incoming variable for which the ratio

   $$c^T(x_{new} - x_{old}) / \| x_{new} - x_{old} \| \text{ is maximum.}$$

4. Bland's rule (guaranteed to prevent cycling)

   Choose the incoming variable with the smallest index.

   Choose the outgoing variable with the smallest index.

5. Random edge

   Choose the incoming variable uniformly from the available possibilities.

The lexicographic rule

Consider the tableau

$$\begin{array}{ll} x_5 = & -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ x_6 = & -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ x_7 = 1 & - x_1 \\ \hline z = & 10x_1 - 57x_2 - 9x_3 - 24x_4 \end{array}$$

Simplex cycles if

(i) The entering variable is chosen to have the largest index.

(ii) The leaving variable is chosen to have the smallest index.

IDEA: Perturb the RHS of the equations

$$\begin{array}{ll} x_5 = \varepsilon_1 & -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ x_6 = \varepsilon_2 & -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ x_7 = 1 + \varepsilon_3 & - x_1 \\ \hline z = & 10x_1 - 57x_2 - 9x_3 - 24x_4 \end{array}$$

Treat $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ as abstract very small numbers.

$$1 \gg \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg 0$$

Add them to the RHS of the m equations. Proceed as before.

$3 - \varepsilon_1$

3

$2 + 10\varepsilon_1$

$3 - 4\varepsilon_1 + \varepsilon_2$

$\varepsilon_2 + \varepsilon_3$

$3 + 4\varepsilon_1 + \varepsilon_3$

$3 - 4\varepsilon_1 + \varepsilon_2 + \varepsilon_3$

$\varepsilon_2 + \varepsilon_3, \quad 2 + 10\varepsilon_1, \quad 3 - 4\varepsilon_1 + \varepsilon_2$

$$
\boxed{\begin{array}{rcl}
x_1 &=& 2\varepsilon_2 + 3x_2 + x_3 - 2x_4 - 2x_6 \\
x_5 &=& \varepsilon_1 - \varepsilon_2 + 4x_2 + 2x_3 - 8x_4 + x_6 \\
x_7 &=& 1 - 2\varepsilon_2 + \varepsilon_3 - 3x_2 - x_3 + 2x_4 + 2x_6 \\
\hline
z &=& 20\varepsilon_2 - 27x_2 + x_3 - 44x_4 - 20x_6
\end{array}}
$$

$$\downarrow$$

OPTIMAL

$$
\boxed{\begin{array}{rcl}
x_1 &=& 1 + \varepsilon_3 \\
x_5 &=& 2 + \varepsilon_1 - 5\varepsilon_2 + 2\varepsilon_3 - 2x_2 - 4x_4 + 5x_6 - 2x_7 \\
x_3 &=& 1 - 2\varepsilon_2 + \varepsilon_3 - 2x_2 - 4x_4 + 5x_6 - x_7 \\
\hline
z &=& 1 + 18\varepsilon_2 + \varepsilon_3 - 30x_2 - 42x_4 - 18x_6 - x_7
\end{array}}
$$

Theorem: The simplex method will not revisit a previously visited basis if the leaving variable is chosen using the lexicographic rule.

$$\Downarrow$$

SIMPLEX TERMINATES

CLAIM: At the end of each pivot step the "scalar term" of the form $r_0 + r_1\varepsilon_1 + r_2\varepsilon_2 + \cdots + r_m\varepsilon_m$ in each row is non-zero.

Proof: In the beginning, the scalar terms have the form

$$\begin{pmatrix} b_1 + \varepsilon_1 \\ b_2 + \varepsilon_2 \\ \vdots \\ b_m + \varepsilon_m \end{pmatrix} = \begin{pmatrix} b_1 & 1 & & & & \\ b_2 & & 1 & & & \\ \vdots & & & 1 & & \\ & & & & 1 & \\ b_m & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{pmatrix}$$

The scalar terms are subsequently modified when we scale an equation, or add a multiple of one equation to another. These operations correspond to multiplying the vector of scalars by a non-singular matrix.

But

$$E \begin{pmatrix} b_1 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ b_m & & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \varepsilon_1 \\ \\ \varepsilon_m \end{pmatrix} = \begin{array}{c} \quad e_1, e_1 \\ \left(\begin{array}{c:c} E_1 b & E \end{array} \right) \end{array} \begin{pmatrix} 1 \\ \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{pmatrix}$$

Since the rows of B are linearly independent, the scalars that appear on the RHS of the tableau are linearly independent polynomials in $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$. $\boxtimes$

CLAIM $\Rightarrow$ The objective function z continuously increases as a polynomial in $\varepsilon_1, \ldots, \varepsilon_m$.

Efficiency of simplex

- In practice simplex seems to require only a small number of iterations.
- There are examples with n variables and n inequalities where simplex takes $\sim 2^n$ iterations if it uses the Dantzig rule.
- Similar examples exist for other rules as well.
- With a randomized strategy, simplex is known to terminate in $\exp(c\sqrt{n \log n})$ steps in expectation.
- For certain probabilistic models for generating LPs, the expected running time of simplex is polynomially bounded in the size of the LP.
- With an appropriate pivot rule simplex algorithm runs in expected polynomial time on worst-case instances with random perturbations.

Smoothed analysis

Duality

The original LP

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Idea: Obtain new equations by taking linear combinations of existing equations.

A typical such equation looks like

$$(y_1\ y_2 \cdots y_m)\, A x = (y_1\ y_2 \cdots y_m)\, b$$

i.e. $$y^T A x = y^T b$$

Suppose we choose y so that

$$c^T = y^T A,$$

then

$$c^T x = y^T A x = y^T b,$$

that is, $$y^T b = \text{Opt}$$

We can say more since $x \geq 0$.

Suppose we can arrange y^T such that

$$c^T \leq y^T A,$$

then

$$c^T x \leq y^T A x = y^T b$$

To obtain the best upper bound, solve the LP.

$$\boxed{\begin{array}{lll} \text{minimize} & b^T y & \text{DUAL} \\ \text{subject to} & A^T y \geq c & \\ & y \in \mathbb{R}^m & \text{non-}\cancel{\text{negativity}} \end{array}}$$

Questions: Is the DUAL feasible?

Is the DUAL unbounded?

Theorem: If both are bounded, then

$$\mathrm{Opt}(\text{PRIMAL}) \leq \mathrm{Opt}(\text{DUAL}).$$

When do we have equality?

SIMPLEX $\Rightarrow$ ALWAYS.