Last time

1. Duality

$$Ax = b \qquad A^T y \geq c$$
$$x \geq 0$$

PRIMAL	DUAL
maximize $c^T x$	minimize $b^T y$
subject to $Ax \leq b$	subject to $A^T y \geq c$
$x \geq 0$	$y \geq 0$

STRONG DUALITY

Primal has an optimal solution $\Rightarrow$ Dual has an optimal solution and OPT(PRIMAL) = OPT(DUAL)

Convert the PRIMAL to equational form and solve it using SIMPLEX.

maximize $c^T x$
subject to $Ax + I x_S = b$
$x, x_S \geq 0$

$$\bar{A} = (\, A \mid I \,)$$

We stop with the tableau:

$$x_B = \bar{A}_B^{-1} b - \bar{A}_N x_N$$
$$z = OPT - r^T x_N$$

Claim: $Z = OPT - r^T x_N = c^T x - y^T (b - \bar{A} x)$

for some $y \geq 0$

(Last time: "This is clear from the way the last line of the tableau is modified in each pivot step.")

Today: What must be y?

In $Z = c^T x + y^T (b - \bar{A} x)$ all basic variables have coefficient zero. So,

$$\boxed{y^T = c_B^T (\bar{A}_B)^{-1}}$$

$$\begin{array}{c} -\left(\bar{A}_B \;\vdots\; \bar{A}_N \right) \\ \left(c_B^T \;\vdots\; c_N^T \right) \end{array}$$

Verify that this y^T is a dual optimum.

$$Z = c^T x = c_B^T x_B + c_N^T x_N \; ; \; x_B = \bar{A}_B^{-1} b - \bar{A}_B^{-1} \bar{A}_N x_N$$

$$\text{OPT(PRIMAL)} = c_B^T \bar{A}_B^{-1} b = b^T (\bar{A}_B^{-1})^T c_B$$
$$= \text{Obj}(y)$$

So y does have the right objective value.

But is y feasible?

$$c^T x = c_B^T x_B + c_N^T x_N = c_B^T(\bar{A}_B^{-1} b - \bar{A}_B^{-1} \bar{A}_N x_N) + c_N^T x_N$$

$$= y^T(b - \bar{A}_N x_N) + c_N^T x_N = c_B^T x_B + y^T(b - \bar{A}x) + c_N^T x_N$$

as expected

$$c^T x + y^T(b - \bar{A}x) = \mathrm{OPT(PRIMAL)} - r^T x$$

$\Downarrow$ compare coefficients

For basic variable x_j : $c_j = (\bar{A}^T)_{j*} y$ ✓

For non-basic variables x_k : $c_k = (\bar{A}^T)_{k*} y - r_k$

For non-basic non-slack x_k : $c_k \leq \bar{A}_{k*} y$ ✓

For slack variables x_{k+n} : $0 \leq y_k$ ✓
(basic or non basic)

$\boxed{\text{So } y \text{ is feasible.}}$

[Recall that our variables are

slack variables

$x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m}$]

The following discussion is based on Schrijver's Theory of linear and integer programming, chapter 7.

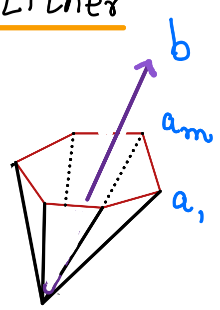
2. The fundamental theorem of linear inequalities

Theorem: $a_1, a_2, \dots, a_n, b \in \mathbb{R}^m$,

$\operatorname{rank}(\{a_1, \dots, a_n, b\}) = m$

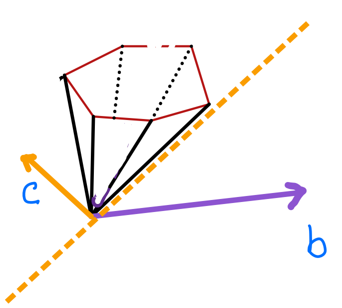
$$A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Either



I. b is a non-negative linear combination of $a_1, \dots, a_n$, i.e., $Ax = b$, $x \geq 0$ is feasible;

Or



II. There is a hyperplane $\{x : c^T x = 0\}$ containing $m-1$ linearly independent vectors from $a_1, a_2, \dots, a_n$ s.t.

$$\left.\begin{array}{l} c^T b < 0 \quad \text{and} \\ c^T a_1, c^T a_2, \dots, c^T a_n \geq 0 \end{array}\right\} *$$

Both I and II cannot hold.

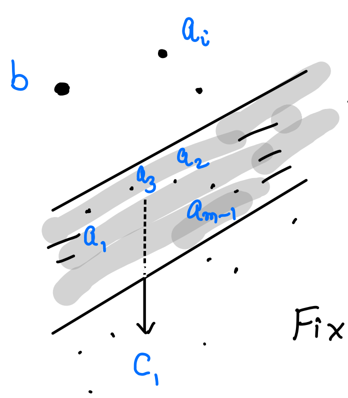
$$Ax = b \quad \Downarrow \quad 0 \le \underbrace{c^T A}x = c^T b$$

$\neg I \Rightarrow II$: The existence of c follows from strong duality, Fourier-Motzkin elimination, etc.

TODAY: A simplex-like proof

- We may assume that $a_1, a_2, \dots, a_n$ span $\mathbb{R}^m$.
  Otherwise, $\exists c$ s.t. $c^T a_i = 0$ for $i = 1, \dots, n$
  $c^T b = -1$
- We may assume $b \neq 0$, otherwise $A\,0 = b$.
- Extend b to a basis $(b, a_1, a_2, \dots, a_{m-1})$, say.

  Let H_1 be the hyperplane passing through $\underbrace{a_1, \dots, a_{m-1}}_{\text{initial anchor points}}$.



$H_1 = \{x : c_1^T x = 0\}$

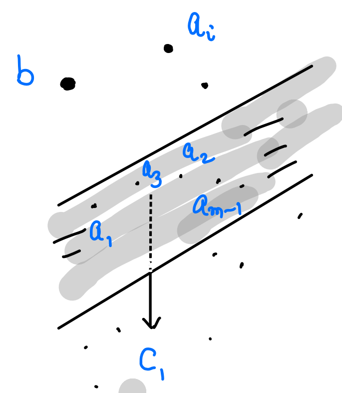
If c_1 is not good, there must be an a_i on the same side as b.

Fix a_i with the smallest index.

BLAND'S RULE

IDEA: Pivot to obtain a new hyperplane where a_i becomes an anchor point replacing one of $a_1, \dots, a_{m-1}$.

Whom to exclude



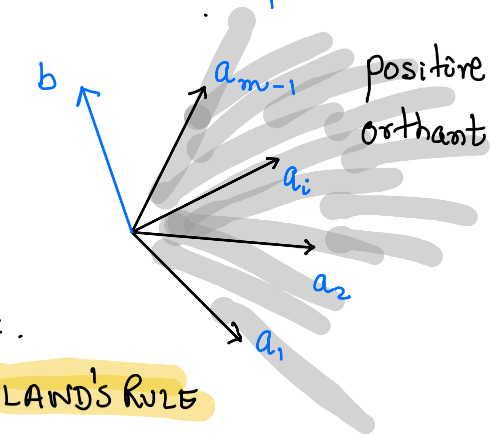
b cannot be in the positive orthant of $(a_1, \ldots, a_{m-1}, a_i)$.

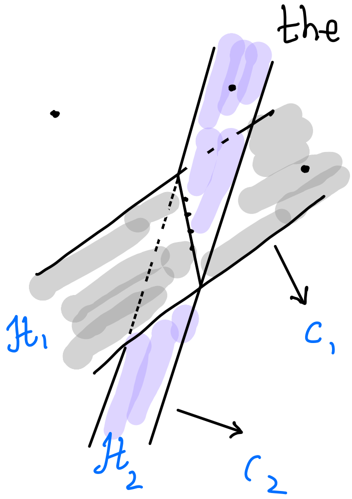
$b = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{m-1} a_{m-1} + \alpha_i a_i$

- $\alpha_i > 0$
- Some other $\alpha_j < 0$.

$\Downarrow$

drop a_j where j is the smallest such index.





$\mathcal{H}_2$ separates b and a_j.

BLAND'S RULE

If $\mathcal{H}_2$ is good, we are done.

Otherwise repeat until some $\mathcal{H}_i$ is good.

Anchor points

A_1
A_2
$\vdots$
A_k
A_{k+1}
$\vdots$
A_ℓ

What if we cycle?

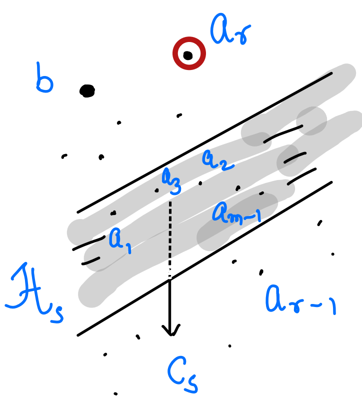
The last fickle element

$a_1, a_2, a_3, \quad , a_{r-1}, a_r, a_{r+1}, \ldots, a_n$

stable

r is the largest index for which a_r enters the anchor set and leaves it later.

- a_r enters A_{s+1}



$c_s^T \cdot b < 0$

$c_s^T a_r < 0$

$c_s^T a_j \geq 0$ for $j = 1, \dots, r-1$

- a_r is replaced in A_t by an earlier fickle element a_j.

$$b = \underbrace{\alpha_1 a_1 + \quad + \alpha_j a_j + \cdots +}_{\text{all } \alpha_j \geq 0} \overset{\alpha_r < 0}{\alpha_r a_r} + \underbrace{\cdots + \alpha_n a_n}_{\text{not fickle}}.$$

Take dot product with c_s

$$0 > c_s^T \cdot b = \underbrace{\alpha_1 c_s^T a_1 + \cdots + \alpha_j c_s^T a_j + \cdots +}_{\text{all positive}} \overbrace{\alpha_r c_s^T a_r}^{\text{both negative}} + \underbrace{\cdots + \alpha_n c_s^T a_n}_{\text{zero}}$$

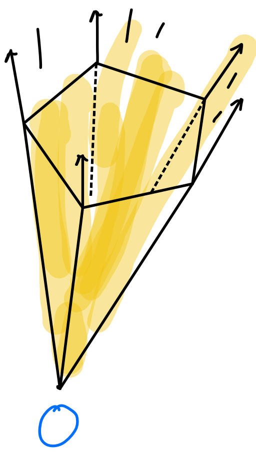
$0 > +\text{ve}$, Contradiction.

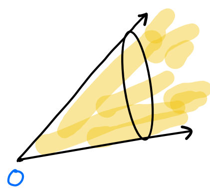
So we must stop.

Cones

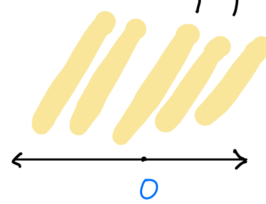
C is a cone if $x, y \in C$ and $\lambda, \mu \geq 0$ then $\lambda x + \mu y \in C$

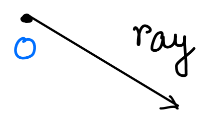
$\mathbb{R}^n$

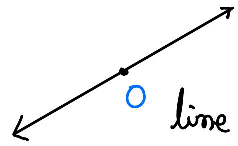




closed half-space







Polyhedral cone $C = \{x : Ax \leq 0\}$ intersection of finitely many closed linear half-spaces.

Finitely generated cone

$$\mathrm{Cone}(\{x_1, x_2, \dots, x_n\}) = \{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n : \lambda_i \geq 0\}$$

A convex polyhedron $P = \{x : Ax \leq b\}$ $\mathbb{R}^k$

$x \in \mathbb{R}^m$, A: $n \times m$

$$\mathrm{Conv.hull}(\{x_1, x_2, \dots, x_n\}) = \left\{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n : \begin{matrix} \lambda_i \geq 0 \\ \sum \lambda_i = 1 \end{matrix}\right\}$$

Theorem (Farkas, Minkowski, Weyl)

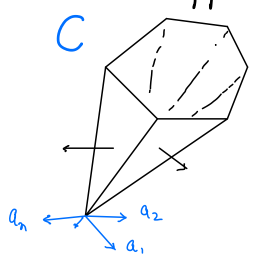
A convex cone is finitely generated

iff

it is polyhedral

$\Rightarrow$ Use the fundamental theorem and list all supporting hyperplanes. $\Rightarrow$ The cone is polyhedral

$\Leftarrow$ Suppose $C = \{x : a_1^T x \le 0,\ a_2^T x \le 0,\ \dots,\ a_n^T x \le 0\}$

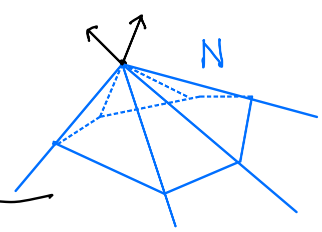


— polyhedral

$$N = \mathrm{Cone}(\{a_1, a_2, \dots, a_n\})$$

$$(\forall a \in N \quad C \subseteq \{x : a^T x \le 0\})$$

finitely generated



outward normals

By the previous part,

$$N = \{x : Bx \le 0\}$$

$$= \{x : b_1^T x \le 0, \dots, b_t^T x \le 0\}$$

Claim: $\mathrm{Cone}(b_1, \dots, b_t) = C \Rightarrow C$ is finitely generated.

- $\text{cone}(b_1, \dots, b_t) \subseteq C$

Every b_i has a non-positive dot product with each element of N.
So each $b_i \in C$.

- Suppose $x \notin \text{cone}(b_1, b_2, \dots, b_t)$.

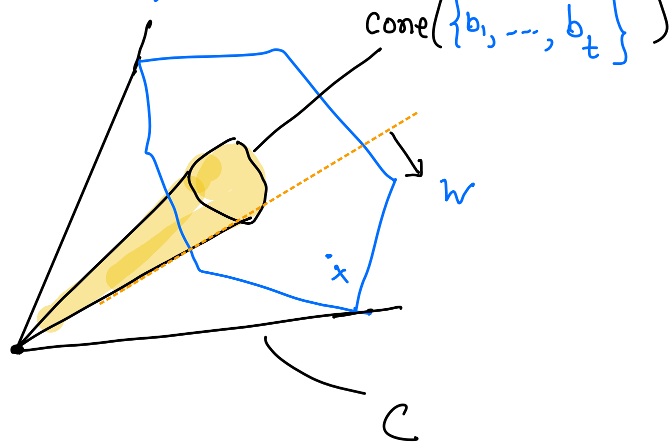
Then, $\exists w$ s.t $w^T b_i \leq 0 \;\forall i$ and $w^T x > 0$.

$$\Downarrow$$

$$w \in N$$

Since N is a set of valid outward normals for C,
we conclude that $x \notin C$.

Key idea: $w \in N$



Decomposition theorem for polyhedra.

P is a polyhedron $\iff$ $P = Q + C$

Q = Convex hull of a finite set, C polyhedral cone

I. Consider

$$\left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : x \in \mathbb{R}^n, \lambda \in \mathbb{R}, \lambda \geq 0, Ax - \lambda b \leq 0 \right\}$$

polyhedral cone finitely generated by $\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix}$

We may assume that each $\lambda_i = 0, 1$.

Set $Q = \mathrm{Conv}(\{x_i : \lambda_i = 1\})$

$C = \mathrm{Cone}(\{x_i : \lambda_i = 0\})$

II Suppose $P = Q + C$, where $Q = \text{conv. hull}(\{x_1, \dots, x_n\})$ and $C = \text{cone}(\{y_1, \dots, y_t\})$

Consider the cone $K = \text{cone}\left\{\begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ 1 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} y_t \\ 0 \end{pmatrix}\right\}$

$$K = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : Ax + \lambda b \leq 0 \right\}$$

$$P = K \cap \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R}^m \right\} = \{ x : Ax \leq -b \}$$

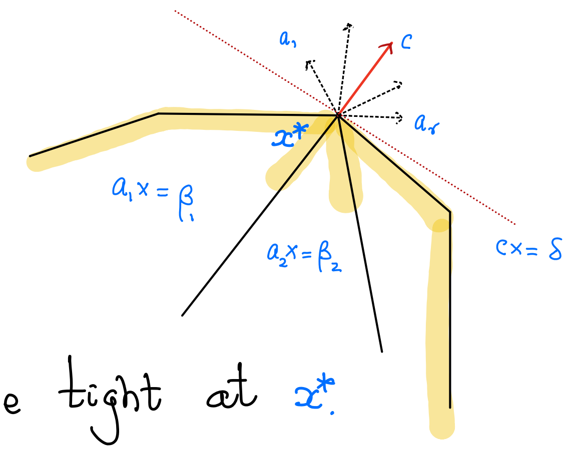
So P is a polyhedron.

Corollary: A set P is a convex hull of a finite set of points iff P is a bounded polyhedron.

The geometry (physics?) of LP-duality

LP maximize $c^T x$

subject to $Ax \leq b$

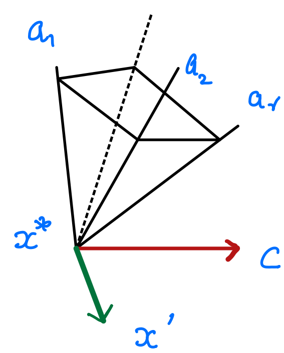


Suppose x^* is an optimum.

Let $a_1 x \leq \beta_1, \ldots, a_r x \leq \beta_r$ be tight at x^*.

Claim:

$c \in \text{cone}(\{a_1, a_2, \ldots, a_r\})$



$x^* + \varepsilon x'$ is feasible for ε small enough.

$c^T(x^* + \varepsilon x') = c^T x^* + \varepsilon c^T x' > c^T x^*$. contradiction

Suppose $\quad c = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_r a_r \qquad (\lambda_i \geq 0)$

Then $\quad c^T x^* = \lambda_1 a_1^T x^* + \lambda_2 a_2^T x^* + \cdots + \lambda_r a_r^T x^*$

$$\begin{aligned} \text{OPT (PRIMAL)} &= c^T x^* = \lambda_1 \beta_1 + \lambda_2 \beta_2 + \cdots + \lambda_r \beta_r \\ &\geq \min_y \{ y^T b : y^T A = c, \ y \geq 0 \} \\ &= \text{OPT (DUAL)} \end{aligned}$$

$\leq$ always holds: $\qquad Ax^* \leq b$

$$c^T x^* = y^T A x^* \leq y^T b$$