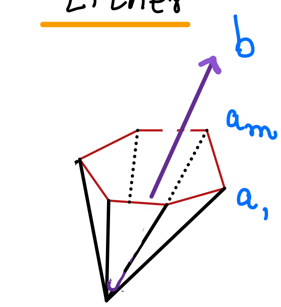
1. The fundamental theorem of linear inequalities

Theorem: $a_1, a_2, \dots, a_n, b \in \mathbb{R}^m$,

$$\operatorname{rank}(\{a_1, \dots, a_n, b\}) = m$$

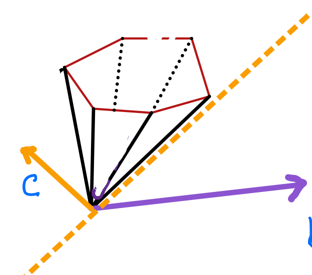
$$A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Either



I. b is a non-negative linear combination of $a_1, \dots, a_n$, i.e., $Ax = b$, $x \geq 0$ is feasible;

Or



II. There is a hyperplane $\{x : c^T x = 0\}$ containing $m-1$ linearly independent vectors from $a_1, a_2, \dots, a_n$ s.t.

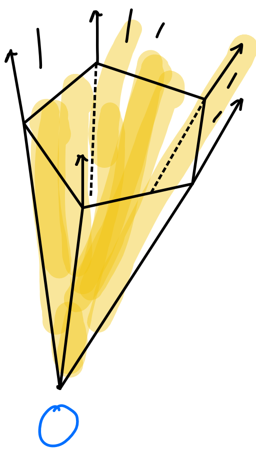
$$\left.\begin{array}{l} c^T b < 0 \quad \text{and} \\ c^T a_1, c^T a_2, \dots, c^T a_n \geq 0 \end{array}\right\} *$$

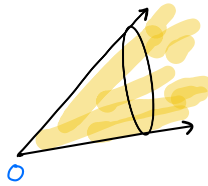
2. Cones

$\mathbb{R}^n$

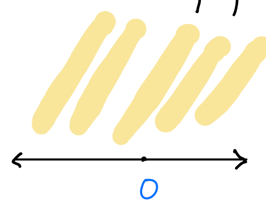
C is a cone if $x, y \in C$ and $\lambda, \mu \geq 0$

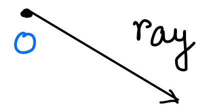
then $\lambda x + \mu y \in C$



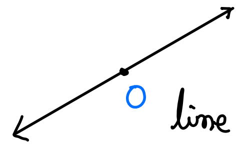


closed half-space





ray



line

Polyhedral cone $C = \{x : Ax \leq 0\}$ intersection of finitely many closed linear half-spaces.

Finitely generated cone

$$\text{Cone}(\{x_1, x_2, \dots, x_n\}) = \{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n : \lambda_i \geq 0\}$$

A convex polyhedron $P = \{\underset{\mathbb{R}^m}{\underset{\cap}{x}} : \underbrace{A}_{n \times m} x \leq \overset{\mathbb{R}^n}{b}\}$

$$\text{Conv. hull}(\{x_1, x_2, \dots, x_n\}) = \left\{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n : \begin{matrix} \lambda_i \geq 0 \\ \sum \lambda_i = 1 \end{matrix}\right\}$$

Theorem (Farkas, Minkowski, Weyl)

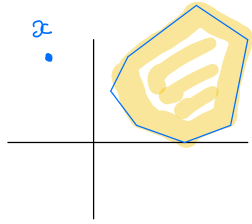
A convex cone is finitely generated
iff
it is polyhedral.

Decomposition theorem for polyhedra.

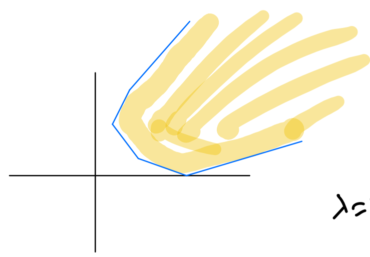
P is a polyhedron $\Longleftrightarrow$ $P = Q + C$

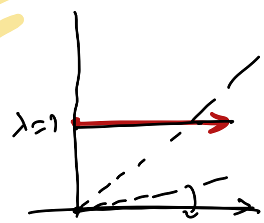
Q = Convex hull of a finite set , C polyhedral cone

I. Suppose P is a polyhedron in $\mathbb{R}^d$.

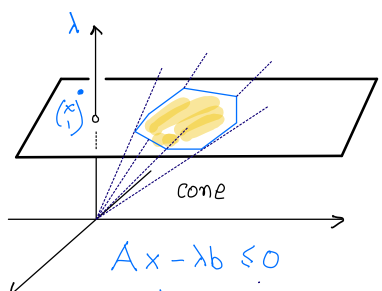


in $\mathbb{R}^d$

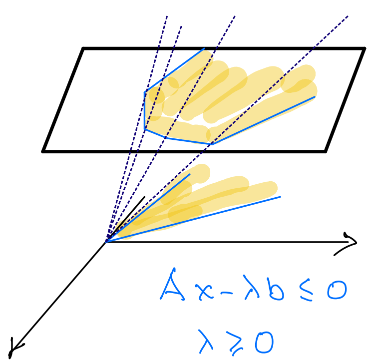




Embed it in $\mathbb{R}^{d+1}$



$Ax - \lambda b \leq 0$
$\lambda \geq 0$



$Ax - \lambda b \leq 0$
$\lambda \geq 0$

Consider

$$\left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : x \in \mathbb{R}^n, \lambda \in \mathbb{R}, \lambda \geq 0, Ax - \lambda b \leq 0 \right\}$$

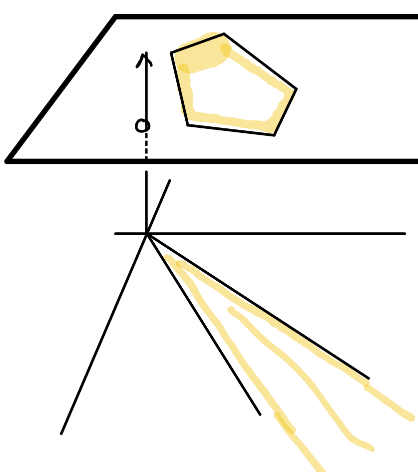
polyhedral cone finitely generated by $\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix}, \dots, \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix}$

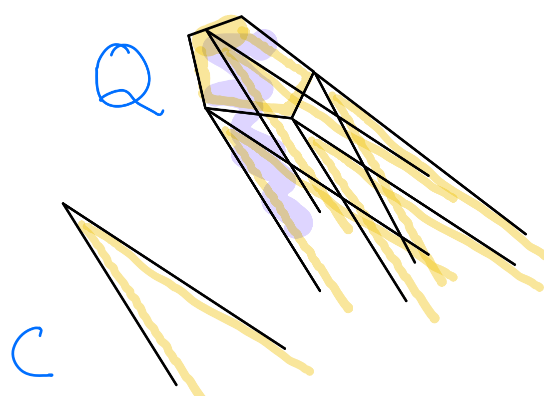
We may assume that each $\lambda_i = 0, 1$.

Set $$\left.\begin{aligned} Q &= \mathrm{Conv}(\{x_i : \lambda_i = 1\}) \\ C &= \mathrm{Cone}(\{x_i : \lambda_i = 0\}) \end{aligned}\right\} \Rightarrow P = Q + C$$

II Suppose $P = Q + C$, where $Q = \mathrm{conv.hull}(\{x_1, \dots, x_n\})$ and $C = \mathrm{Cone}(\{y_1, \dots, y_t\})$.

Consider the cone $K = \mathrm{Cone}\left\{ \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ 1 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} y_t \\ 0 \end{pmatrix} \right\}$





$$K = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : \; Ax + \lambda b \leq 0 \right\}$$

$$P = \left\{ x : \begin{pmatrix} x \\ 1 \end{pmatrix} \in K \right\} = \left\{ x : Ax \leq -b \right\}$$

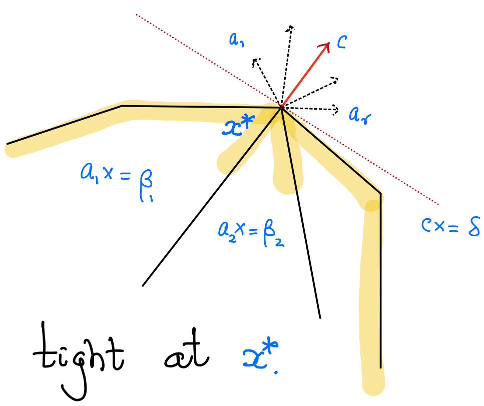
So P is a polyhedron.

Main Theorem of Polytope Theory

A set P is a convex hull of a finite set of points iff P is a bounded polyhedron.

The geometry (physics?) of LP-duality

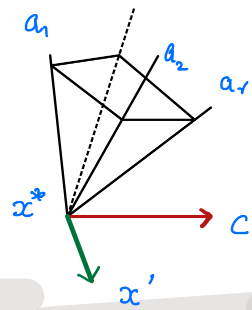
LP maximize $c^T x$
subject to $Ax \leq b$



Suppose x^* is an optimum.

Let $a_1 x \leq \beta_1, \ldots, a_r x \leq \beta_r$ be tight at x^*.

Claim: $c \in \text{cone}(\{a_1, a_2, \ldots, a_r\})$



$x^* + \varepsilon x'$ is feasible for ε small enough.

$c^T(x^* + \varepsilon x') = c^T x^* + \varepsilon c^T x' > c^T x^*$. contradiction

Suppose $c = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_r a_r \quad (\lambda_i \geq 0)$

Then $c^T x^* = \lambda_1 a_1^T x^* + \lambda_2 a_2^T x^* + \cdots + \lambda_r a_r^T x^*$

$$\begin{aligned} \text{OPT(PRIMAL)} &= c^T x^* = \lambda_1 \beta_1 + \lambda_2 \beta_2 + \cdots + \lambda_r \beta_r \\ &\geq \min_y \{ y^T b : y^T A = c^T, y \geq 0 \} \\ &= \text{OPT(DUAL)} \end{aligned}$$

$\leq$ always holds: $\quad Ax^* \leq b$

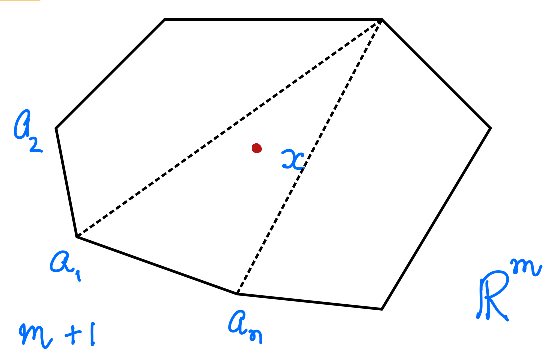
$$c^T x^* = y^T A x^* \leq y^T b$$

Carathéodory's Theorem (HW)

$x \in \text{Conv}(a_1, a_2, \ldots, a_n)$

$\Downarrow$

$x \in \text{Conv}(a_i : i \in S), \; S \subseteq [n], \; |S| \leq m+1$



Today: Colourful Carathéodory Theorem

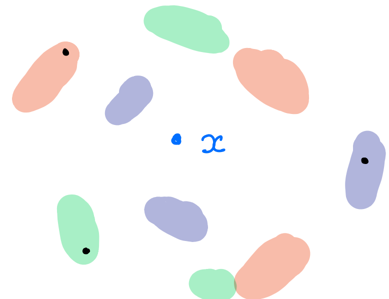
$$S_1, S_2, \ldots, S_{d+1} \subseteq \mathbb{R}^d$$

$$x \in \operatorname{conv}(S_1) \cap \operatorname{conv}(S_2) \cap \ldots \cap \operatorname{conv}(S_{d+1})$$

$$\Downarrow$$

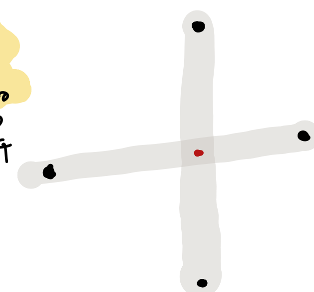
$$\exists x_1 \in S_1, \; x_2 \in S_2, \; \ldots, \; x_{d+1} \in S_{d+1}$$

s.t. $\quad x \in \operatorname{conv}(\{x_1, x_2, \ldots, x_{d+1}\})$



Proof later.

Radon's theorem



$$x_1, x_2, \ldots, x_{d+1}, x_{d+2} \in \mathbb{R}^d$$

$$\Downarrow$$

$$\exists S \subseteq [d+2] \quad \operatorname{conv}(\{x_i : i \in S\}) \cap \operatorname{conv}(\{x_i : i \in \bar{S}\}) \neq \emptyset$$

Proof: $\begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \begin{pmatrix} x_2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_{d+2} \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}$ are linearly dependent.

$$\lambda_1 \begin{pmatrix} x_1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} x_2 \\ 1 \end{pmatrix} + \cdots + \lambda_{d+2} \begin{pmatrix} x_{d+2} \\ 1 \end{pmatrix} = 0, \quad \exists i \; \lambda_i \neq 0$$

Let $S = \{ i : \lambda_i > 0 \}$.

$$\sum_{i \in S} \lambda_i \binom{x_i}{1} = \sum_{j \in \bar{S}} (-\lambda_j) \binom{x_j}{1}$$

$$\Downarrow$$

$$\sum_{i \in S} \lambda_i x_i = \sum_{j \in \bar{S}} (-\lambda_j) x_j \quad \text{and} \quad \underbrace{\sum_{i \in S} \lambda_i}_{= L} = \sum_{j \in \bar{S}} (-\lambda_j)$$

$$\underbrace{\sum_{i \in S} \left(\frac{\lambda_i}{L}\right) x_i}_{\in\, \mathrm{conv}(\{x_i : i \in S\})} = \underbrace{\sum_{j \in \bar{S}} \left(\frac{-\lambda_j}{L}\right) x_j}_{\in\, \mathrm{conv}(\{x_j : j \in \bar{S}\})}$$

Tverberg's theorem

$$\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m \in \mathbb{R}^d, \qquad m = (d+1)(n-1)+1$$

$$\Downarrow$$

A partition $T_1 \cup T_2 \cup \ldots \cup T_n = [m]$ such that

$$\mathrm{conv}(\{\hat{x}_i : i \in T_1\}) \cap \mathrm{conv}(\{\hat{x}_i : i \in T_2\}) \cap \ldots \cap \mathrm{conv}(\{\hat{x}_i : i \in T_n\}) \neq \emptyset$$

Work in a vector space of dimension $(d+1)(n-1)$. ← How?

1. Set $x_i = \binom{\hat{x}_i}{1} \in \mathbb{R}^{d+1}$

2. Pick n sign vectors in $\mathbb{R}^{n-1}$, $\omega_1, \omega_2, \ldots, \omega_n \in \mathbb{R}^{n-1}$, e.g.,

$$\omega_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \ \omega_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \ \ldots, \ \omega_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}, \ \omega_n = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

Our space $\mathbb{R}^{(d+1)\times(n-1)}$

We are ready for colourful Carathéodory theorem.

$$\underbrace{\begin{matrix} x_1 w_1^T \\ x_1 w_2^T \\ \vdots \\ x_1 w_n^T \end{matrix}}_{S_1} \qquad \underbrace{\begin{matrix} x_2 w_1^T \\ x_2 w_2^T \\ \vdots \\ x_2 w_n^T \end{matrix}}_{S_2} \qquad \cdots \qquad \underbrace{\begin{matrix} x_m w_1^T \\ x_m w_2^T \\ \vdots \\ x_m w_n^T \end{matrix}}_{S_m}$$

which we may view as a long vector with $(d+1)(n-1)$ components.

- Each point is a $(d+1)\times(n-1)$ matrix.
- m is one more than the dimension of the space.

$$\boxed{0 \in \operatorname{conv}(S_1) \cap \operatorname{conv}(S_2) \cap \cdots \cap \operatorname{conv}(S_m)} \quad \text{(why?)}$$

Colourful Carathéodory $\Rightarrow$ colourful convex combination

$$\lambda_1 x_1 w_{i_1}^T + \lambda_2 x_2 w_{i_2}^T + \cdots + \lambda_m x_m w_{i_m}^T = 0$$

$$T_1 = \{ j : w_{i_j} = w_1 \}, \quad T_2 = \{ j : w_{i_j} = w_2 \}, \ \ldots, \ T_n = \{ j : w_{i_j} = w_n \}$$

- The T_i are pairwise disjoint.
- $T_1 \cup T_2 \cup \cdots \cup T_n = [m]$.

Claim: The above $T_1, T_2, \dots, T_n$ meet our requirement.

Let $u_j = \sum_{i \in T_j} \lambda_i x_i$; $\hat{u}_j = \sum_{i \in T_j} \lambda_i \hat{x}_i$ $L_j = \sum_{i \in T_j} \lambda_i$

We will show (i) $\hat{u}_1 = \hat{u}_2 = \dots = \hat{u}_n$

(ii) $L_1 = L_2 = \dots = L_n$ $\Big\} \Rightarrow$ CLAIM ✓

(i)

Now $\boxed{u_1 \omega_1^T} + \boxed{u_2 \omega_2^T} + \dots + \boxed{u_n \omega_n^T} = 0$

$\Rightarrow$ For each unit vector e_k

$$(e_k^T u_1)\omega_1^T + (e_k^T u_2)\omega_2^T + \dots + (e_k^T u_n)\omega_n^T = 0$$

(!)

Because $\Sigma \omega_i = 0$ is the only dependency among the ω_i,

$$e_k^T u_1 = e_k^T u_2 = \dots = e_k^T u_n \quad (k = 1, 2, \dots, d+1)$$

$$\Rightarrow \quad u_1 = u_2 = \dots = u_n \Rightarrow \hat{u}_1 = \hat{u}_2 = \dots = \hat{u}_n$$

(ii) The normalization factor $L_j = \sum_{i \in T_j} \lambda_i$ are all the same.

$\qquad \qquad$ $\| \; L$

Look at the last coordinate of

$x_i = \begin{pmatrix} \hat{x}_i \\ 1 \end{pmatrix}$.

$$\Rightarrow \quad \sum_{i \in T_1} \left(\frac{\lambda_i}{L}\right) \hat{x}_i = \sum_{i \in T_2} \left(\frac{\lambda_i}{L}\right) \hat{x}_i = \dots = \sum_{i \in T_n} \left(\frac{\lambda_i}{L}\right) \hat{x}_i .$$

☺

Proof of the colourful Carathéodory theorem

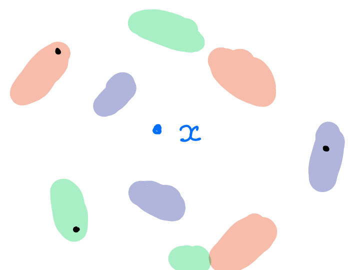
$S_1, S_2, \dots, S_{d+1} \subseteq \mathbb{R}^d$

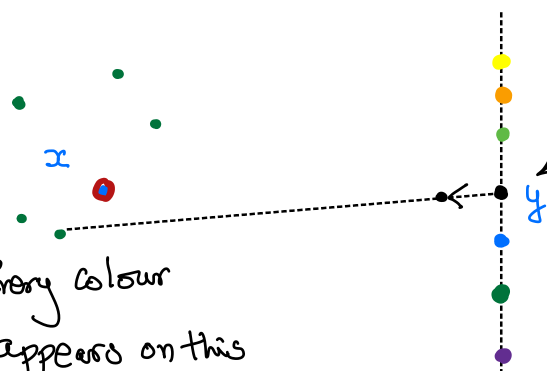
$x \in \mathrm{Conv}(S_1) \cap \mathrm{conv}(S_2) \cap \dots \cap \mathrm{conv}(S_{d+1})$

$$\Downarrow$$

$\exists\, x_1 \in S_1,\ x_2 \in S_2,\ \dots,\ x_{d+1} \in S_{d+1}$

s.t. $\quad x \in \mathrm{Conv}(\{x_1, x_2, \dots, x_{d+1}\})$





closest colourful convex combination

The coloured points that generate y must all lie on a $d-1$ dimensional subspace. By standard Carathéodory some COLOUR is not needed.

Every colour appears on this side.