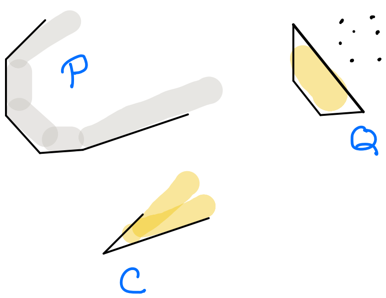
Last time

Decomposition of polyhedra
Radon's theorem
Tverberg's theorem
Colourful Caratheodory theorem

Uniqueness of decomposition

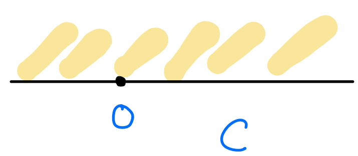
$P = Q + C$

Q: polytope, C: cone



- C is unique, and is called the characteristic cone (or recession cone of P).
- Q is not uniquely determined by P.

not unique!



If $Ax \leq b$ defines P, then $Ax \leq 0$ defines C. So it is unique.

Q is uniquely determined if C is pointed.

pointed: contains no linear subspace other than $\{0\}$.

Theorem: Every polyhedron P has a unique minimal representation of the form

$$P = \text{Conv.hull}(\{x_1, x_2, \ldots, x_r\}) + \text{Cone}(\{y_1, y_2, \ldots, y_t\}) + \text{lin.space}(P)$$

where

- $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r$ are orthogonal to $\text{lin.space}(P) = \ker(A)$,
- $x_1, x_2, \ldots, x_r$ are unique,

and

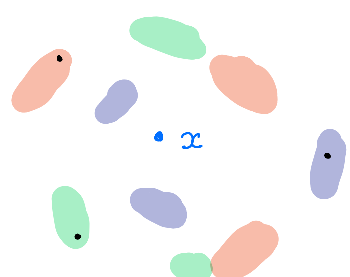
- $y_1, y_2, \ldots, y_t$ are unique up to multiplication by a posive constant.

Colourful Carathéodory Theorem

$$\boxed{\begin{array}{l} S_1, S_2, \ldots, S_{d+1} \subseteq \mathbb{R}^d \\ x \in \text{Conv}(S_1) \cap \text{Conv}(S_2) \cap \ldots \cap \text{Conv}(S_{d+1}) \end{array}}$$

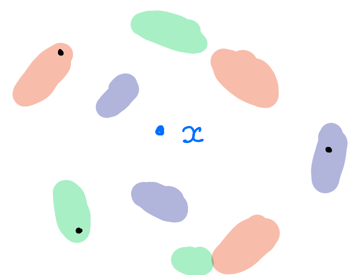
$$\Downarrow$$

$$\boxed{\begin{array}{l} \exists x_1 \in S_1, \; x_2 \in S_2, \ldots, x_{d+1} \in S_{d+1} \\ \text{s.t.} \quad x \in \text{Conv}(\{x_1, x_2, \ldots, x_{d+1}\}) \end{array}}$$



Proof of the colourful Carathéodory theorem

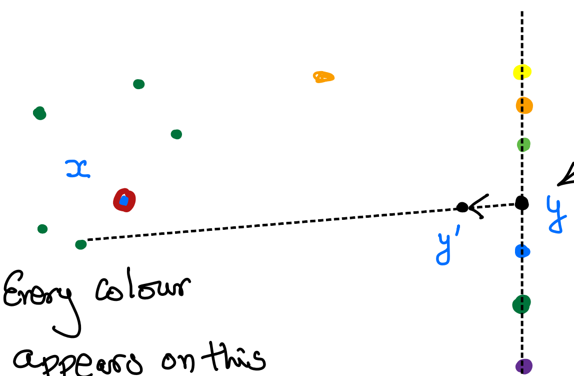
Suppose x cannot be expressed as a colourful convex combination



Let y be a colourful convex combination closest to x.

Then, there is an even closer colourful convex combination.

$\mathbb{R}^d$



closest colourful convex combination

The coloured points that generate y must all lie on a $d-1$ dimensional subspace. By standard Carathéodory some COLOUR is not needed.

Every colour appears on this side.

$\mathbb{R}^{d-1}$

The ellipsoid method

(1) To find an optimal solution of the LP

maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$

it is enough to find a solution to the system

$Ax \leq b$, $x \geq 0$, $A^t y \geq c$, $y \geq 0$, $c^T x \geq b^T y$

(2) To find a solution to a system $Ax = b$, $x \geq 0$, it is enough to be able to determine if such systems have a solution.

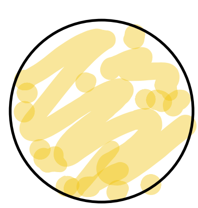
(Repeatedly drop a variable and see if the system is still feasible. The resulting system has a unique solution.)

Ellipsoids

Consider the unit ball in $\mathbb{R}^n$ centered at 0.

$$B(0,1) = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_n^2 \leq 1\}$$

$$= \{x \in \mathbb{R}^n : x^T x \leq 1\}$$



$A^{-1}(y-c)$

Let $A \in \mathbb{R}^{n}$ be a non-singular matrix; $c \in \mathbb{R}^{n}$

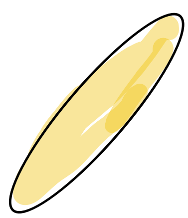
Consider the affine transformation $x \mapsto Ax + c$

The image of the ball under this transformation is

$$\{ y : \ y = Ax + c \ \text{and} \ x \in B(0,1) \}$$

$$= \{ y : \ (y-c)^T \underbrace{(A^{-1})^T A^{-1}}_{\text{symmetric, positive definite}} (y-c) \leq 1 \}$$

Such an object is called an ellipsoid.



D is positive definite

$\Updownarrow$

$D = B^T B$ for some non-singular matrix B

$\Updownarrow$

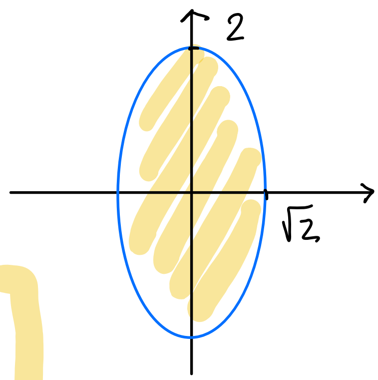
$x^T D x > 0$ for all $x \neq 0$.

So an ellipsoid centered at c is a set of the form

$$\text{ell}(c, D) = \{ x \in \mathbb{R}^n : \ (x-c)^T D^{-1} (x-c) \leq 1 \}$$

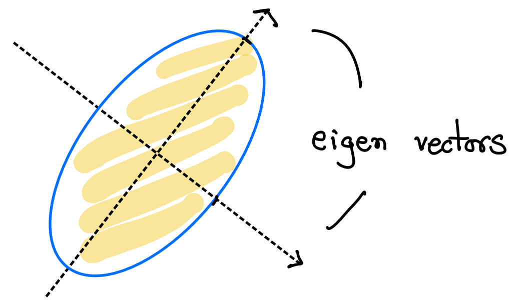
D positive definite $\Leftrightarrow$ $\begin{cases} D \text{ has an orthonormal basis of eigenvectors} \\ \text{All eigen values of } D \text{ are positive.} \end{cases}$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$



$$\boxed{(x_1 \;\; x_2) \begin{pmatrix} 1/2 & \\ & 1/4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 1}$$

$$D = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$



The outline

Assume: (i) The polyhedron $P := \{x : Ax \leq b\}$ is bounded and full-dimensional.

(ii) Calculations can be done precisely.

$A \in \mathbb{R}^{m \times n}$

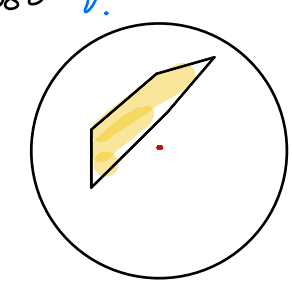
$b \in \mathbb{R}^{n}$

Let $\nu = 4n^2\varphi$, where φ is the maximum size (in bits) of the matrix $[A \mid b]$.

Fact: Each vertex of P has size at most ν.

Initial radius: $R = 2^{\nu}$

$$P \subseteq B(0, R)$$



Khachian: Determine a sequence of ellipsoids

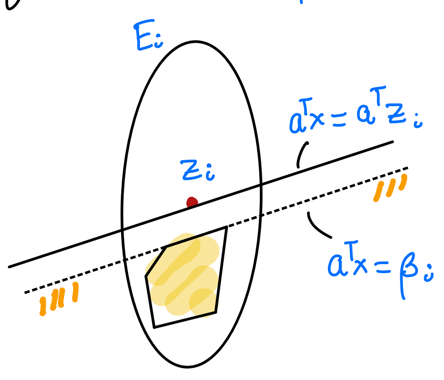
$$E_0, E_1, E_2, \dots, E_i, \dots \quad \text{s.t.} \quad P \subseteq E_i$$

$$B(0, R) = E(z_0, D_0), \quad z_0 = 0, \quad D_0 = R^2 \cdot I \qquad \qquad E_i = E(z_i, D_i)$$

Iteration: Suppose z_i and D_i have been found such that

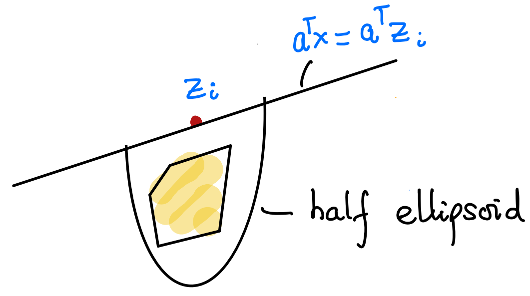
$$P \subseteq E(z_i, D_i)$$

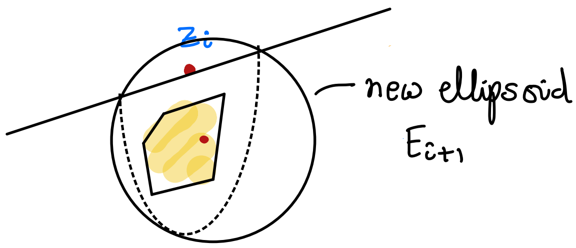
- If $z_i \in P$, we have found a feasible solution. STOP.
- If $z_i \notin P$, then it violates an inequality of the form $a^T x \leq \beta$ in $Ax \leq b$.



Locate such an inequality and consider

$$E_i \cap \{x : a^T x \leq \underbrace{a^T z_i}_{\text{scalar}}\}$$





Claim 1: $$\frac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-1/(2n+2)}$$

Claim 2: $$\mathrm{vol}(P) \geq 2^{-2n\nu}$$

Suppose $N = 16n^2\nu$ iterations are performed.

$$\left.\begin{array}{c}\text{Claim 1}\\ +\\ \text{Claim 2}\end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2^{-2n\nu} \leq \mathrm{vol}(P) \leq \mathrm{vol}(E_N) \\ \qquad\qquad\qquad < e^{-N/(2n+2)}(2R)^n \\ \qquad\qquad\qquad \leq e^{-16n^2\nu/(2n+2)}(2 \cdot 2^{\nu})^n \\ \qquad\qquad\qquad \leq 2^{-2n\nu} \quad ! \end{array}\right.$$

Proof of Claim 2: P is full dimensional

$$\Downarrow$$

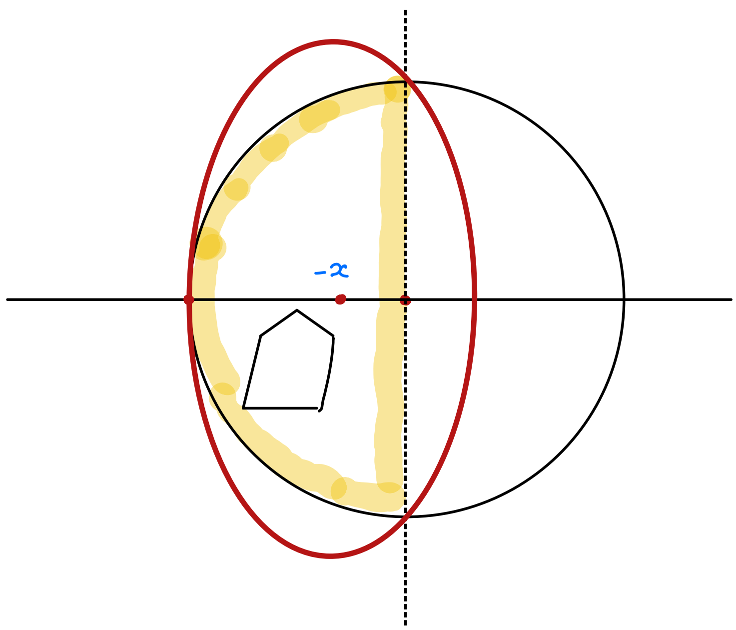
$\exists\, x_0, x_1, \ldots, x_n$ affinely independent

$$\mathrm{vol}(\mathrm{conv.hull}(\{x_0, \ldots, x_n\})) = \frac{1}{n!}\left|\det\left(\begin{bmatrix} 1 & 1 & & 1 \\ x_0 & x_1 & & x_n \end{bmatrix}\right)\right|$$

note that ν is an upper bound on the total length of each x_i.

$$\geq n^{-n}\, 2^{-n\nu} \geq 2^{-2n\nu}.$$

Proof of CLAIM 1: Consider the special case.



$$D_{new} = \begin{pmatrix} \lambda_1^2 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{pmatrix}$$

$$\lambda_1^2 = (1-x)^2$$

$$\frac{x^2}{(1-x)^2} + \frac{1}{\lambda_2^2} = 1$$

$$\Downarrow$$

$$\lambda_2^2 = \frac{(1-x)^2}{1-2x}$$

$$\frac{\mathrm{vol}(\mathbf{new})}{\mathrm{vol}(\mathbf{old})} = \lambda_1 \lambda_2 \cdots \lambda_n$$

$$= \frac{(1-x)^n}{(1-2x)^{(n-1)/2}}$$

to minimize this choose

$$x = \frac{1}{n+1}.$$

For $x = \frac{1}{n+1}$, $\quad \frac{\mathrm{vol}(\mathbf{new})}{\mathrm{vol}(\mathrm{old})} = \frac{\left(n/(n+1)\right)^n}{\left((n-1)/(n+1)\right)^{(n-1)/2}}$

$$= \left(\frac{n}{n+1}\right)\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}}$$

$$\leq \left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n^2-1}\right)^{\frac{n-1}{2}} \leq \exp\left(\frac{-1}{n+1}\right)\exp\left(\frac{n-1}{2(n+1)}\right)$$

$$= \exp\left(\frac{-1}{2(n+1)}\right)$$