Counting Binary Trees
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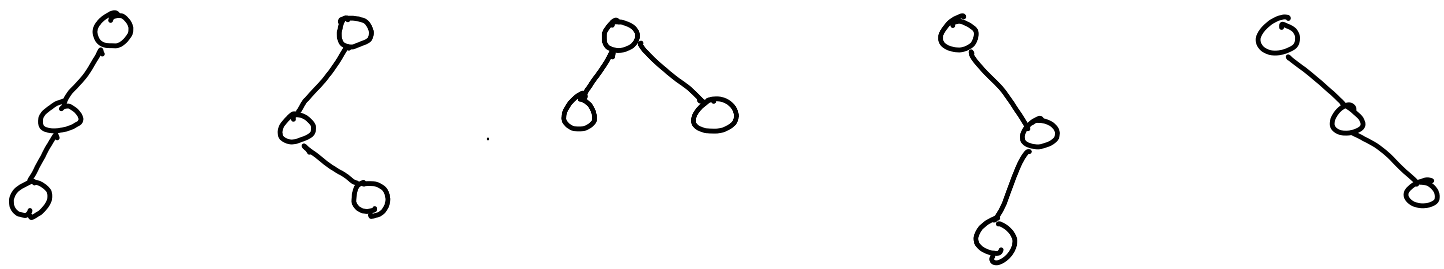
$B_n \triangleq$ # Binary trees (rooted) with n nodes

Ex. Binary trees with 3 nodes :-

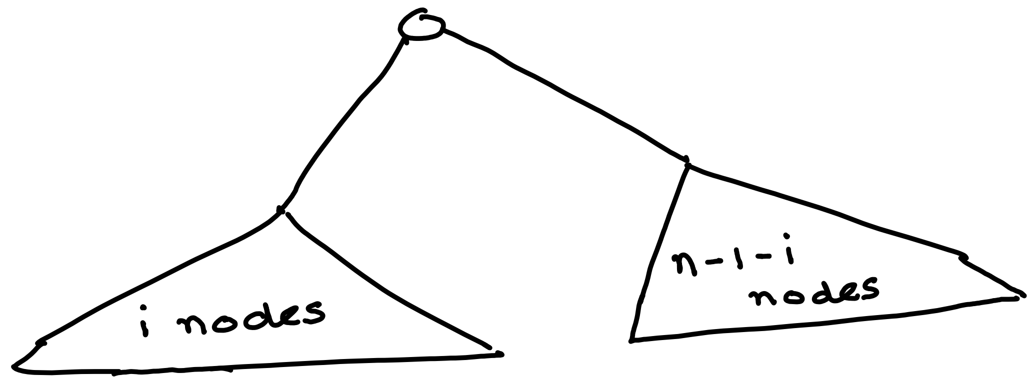


$\therefore \quad B_3 = 5$

Recurrence relation for B_n

$$B_n = B_0 B_{n-1} + B_1 B_{n-2} + \cdots + B_{n-1} B_0$$

$$= \sum_{i=0}^{n-1} B_i B_{n-1-i}$$



$B_i B_{n-1-i}$ such trees

Solving the Recurrence

Generating function for B_n

$$C(x) = B_0 + B_1 x + B_2 x^2 + \cdots$$
$$= \sum_{i=0}^{\infty} B_i x^i$$

$$C(x) \cdot C(x) = (B_0 + B_1 x + B_2 x^2 + \cdots)(B_0 + B_1 x + B_2 x^2 + \cdots)$$
$$= B_0^2 + (B_0 B_1 + B_1 B_0)x + (B_0 B_2 + B_1 B_1 + B_2 B_1)x^2 + \cdots$$
$$= B_1 + B_2 x + B_3 x^2 + \cdots$$

$$\therefore \quad C(x) = 1 + x(C(x))^2$$

$$x\left(C(x)\right)^2 - C(x) + 1 = 0$$

$$\Rightarrow \quad C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Take $\quad C(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}$$

Use Taylor series
(a little hairy)
OR
Look at wikipedia
entry for
Catalan Number

$$\therefore \quad B_n = \frac{1}{n+1}\binom{2n}{n}$$

$P_n \triangleq$ # Balanced parenthesized expressions with n pairs of parentheses.

Characterization of Balanced parentheses :-

i) # left parentheses = # right parentheses

<u>And</u> ii) In any prefix,

left parentheses $\geq$ # right parentheses

Ex. Balanced expressions with 3 pairs of parentheses

((()))   (()())   (())()   ()(())   ()()()

$$P_n = B_n$$

First pair of parentheses

$$(\underbrace{\qquad\qquad\qquad}_{P_i \text{ pairs}})\ \underbrace{\qquad\qquad\qquad}_{P_{n-i-1} \text{ pairs}}$$

$P_i P_{n-i-1}$ such balanced parenthesized expressions

$$P_n = P_0 P_{n-1} + P_1 P_{n-2} + \cdots + P_{n-1} P_0$$

$$= \sum_{i=0}^{n-1} P_i P_{n-i-1}$$

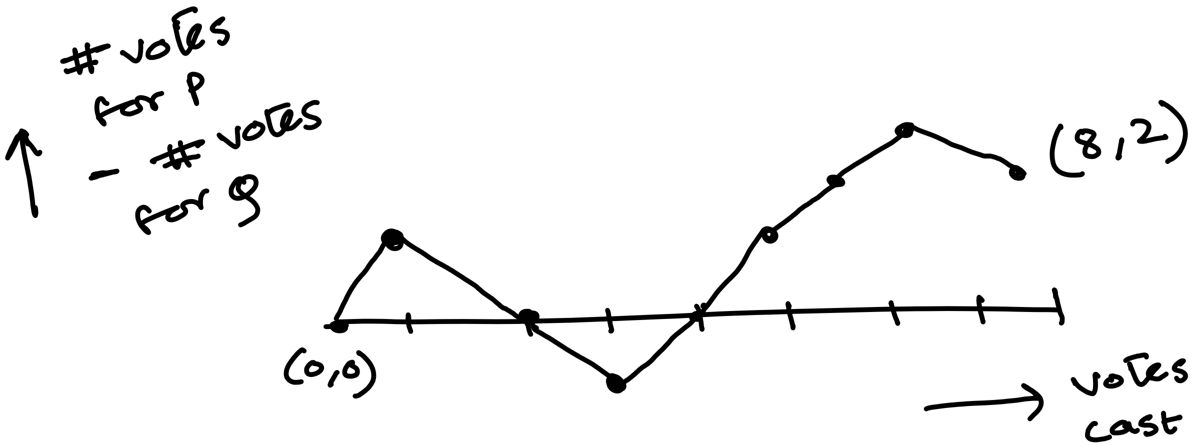
$\because$ $P_0 = B_0$, $P_1 = B_1$ & the recurrence for P_n & B_n are the same, $P_n = B_n$ $\square$

Ballot problem

Suppose in an election, Candidate P gets p votes and candidate Q gets q votes, How many voting patterns are possible, such that Q is never ahead of P.

Represent Voting pattern as a path

Ex. p gets 5 votes, q gets 3 votes

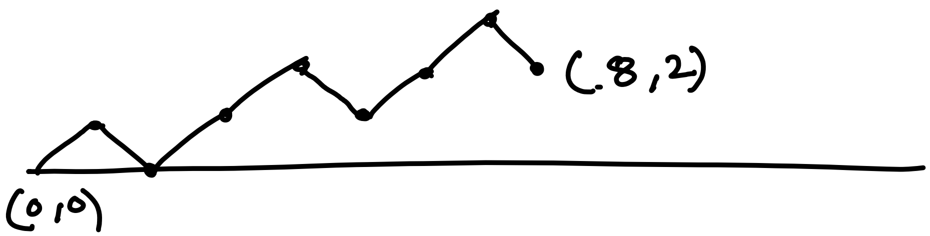


1st vote p
2nd vote q
3rd vote q
4th vote p
5th vote p
6th vote p
7th vote p
8th vote q

Total # paths $= \binom{8}{3} = \binom{8}{5}$

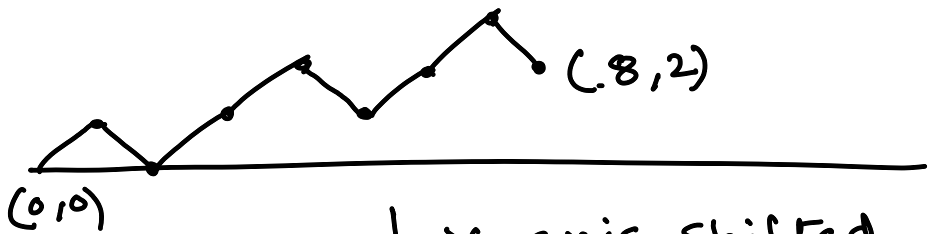
since 8 votes cast and 3 votes are for q

Paths of Interest : when Q is never ahead of P.

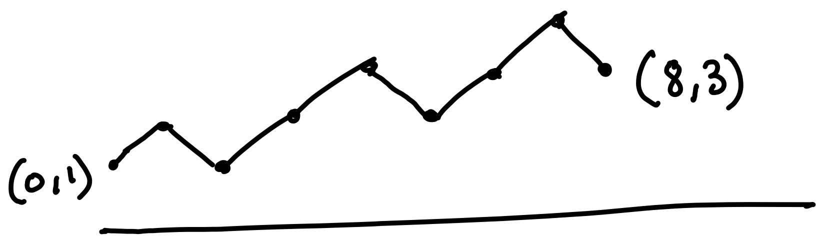


All paths from (0,0) to (8,2) such that the path never goes below the x axis

Shift the X axis 1 below



↓ x axis shifted



All paths from (0,0) to (8,2) such that the path never goes below the x axis

↓ x axis shifted

All paths from (0,1) to (8,3) such that the path never touches the x axis

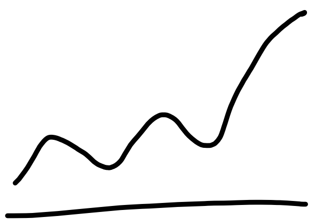
Strategy : Complementation

Good paths = Total # paths − # Bad paths

$$\binom{p+q}{p}$$

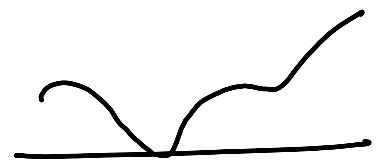
Good path

Ex

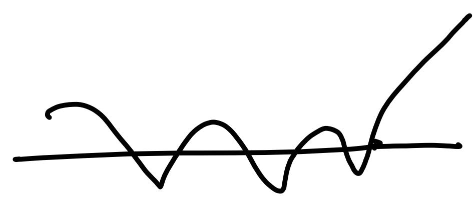


Bad path

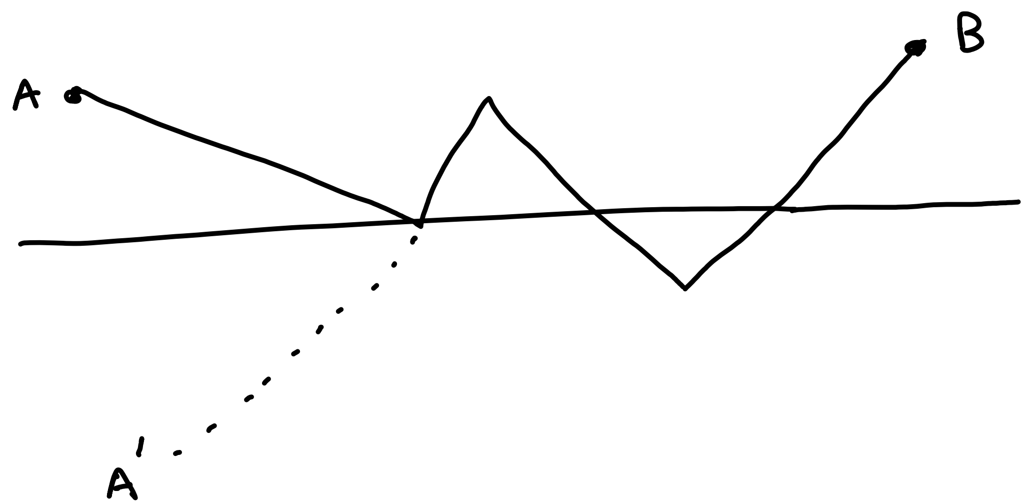
Ex.



Ex

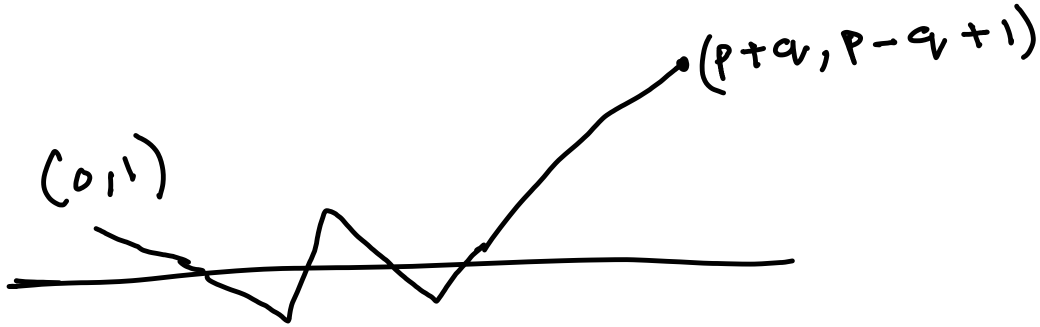


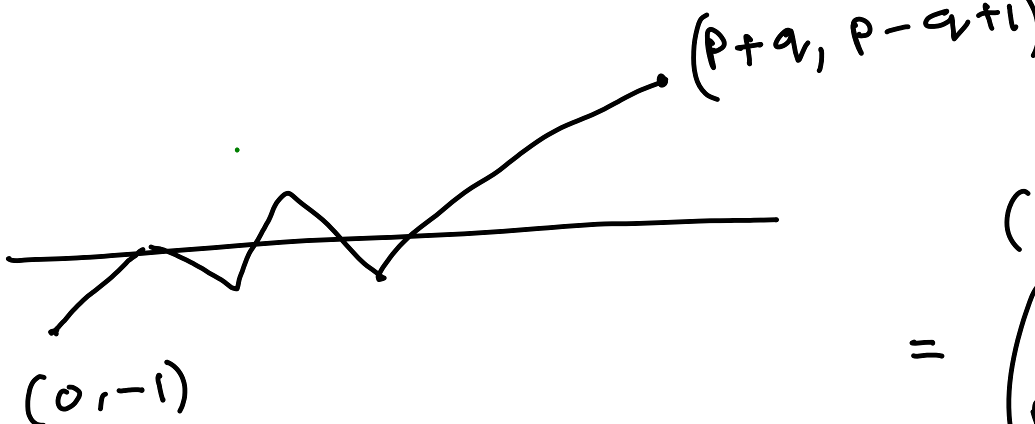
Reflection Principle



Number of paths from A to B that touches or crosses the x axis is same as the number of paths from A' to B.

Count bad paths using Reflection principle





All paths from $(0,-1)$ to $(p+q, p-q+1)$

$$= \binom{p+q}{\frac{p+q-(p-q+2)}{2}}$$

$$= \binom{p+q}{q-1}$$

$\therefore$ # Good Paths $= \binom{p+q}{p} - \binom{p+q}{q-1}$

when $p = q$, $B_n =$ # Good paths

$$= \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

$\boxtimes$

Thank You for your attention!