On Joint Triangulations of Two Sets of Points in the Plane

Subir Kumar Ghosh

School of Technology & Computer Science
Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai 400005
ghosh@tifr.res.in
Jointly with

1. Ajit Arvind Diwan, Department of Computer Science and Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India
2. Partha Pratim Goswami, Institute of Radiophysics and Electronics, University of Calcutta, Kolkata 700009, India
3. Andrzej Lingas, Department of Computer Science, Lund University, S-221 00 Lund, Sweden
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Theorem: A set of $n$ points in the plane can be triangulated in $O(n \log n)$ time.

Joint triangulation or Compatible triangulation

Consider two sets $A$ and $B$ of points in the plane, where $|A| = |B| = n$. 

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\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
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Two triangulations $T_a$ of $A$ and $T_b$ of $B$ are called joint triangulation (also called compatible triangulation) of $A$ and $B$ if there exists a bijection $f$ between $A$ and $B$ such that (i) $ijk$ is a triangle in $T_a$ if and only if $f(i)f(j)f(k)$ is a triangle in $T_b$, and (ii) $ijk$ and $f(i)f(j)f(k)$ do not contain any point of $A$ and $B$ respectively.
The problem has applications in morphing and automated cartography.

The problem of joint triangulation of $A$ and $B$ has two variations depending upon whether the bijection between points of $A$ and $B$ are fixed in advance.

Here, we consider the problem, where the bijection is fixed in advance.


Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two disjoint sets of points in the plane, specified by their respective $x$ and $y$ coordinates.
Find triangulations of $A$ and $B$ (say, $T(A)$ and $T(B)$) such that for each region bounded by a triangle $a_iaジャa_k$ in $T(A)$, the corresponding triangle $b_iジャb_k$ bounds a region in $T(B)$. 
Background

The problem was posed in 1987 by Saalfeld.

Over the last two decades, several researchers have worked on this problem but the problem is still open.

Is it a joint triangulation?

- For every line segment $a_i a_j$ in $T(A)$, the corresponding line segment $b_i b_j$ is in $T(B)$ and vice versa.
- However, the triangle $a_4 a_5 a_6$ does not contain any point of $A$, whereas the corresponding triangle $b_4 b_5 b_6$ contains points of $B$.
- Thus the triangles bounding the regions are different and we do not consider this to be a joint triangulation.
A triangle in $T(A)$ or $T(B)$ is said to be a \textit{component triangle} of the triangulation if it does not contain any point in its interior.

The problem of joint triangulation of $A$ and $B$ is to compute $T(A)$ and $T(B)$, if they exist, such that a triangle $a_ia_ja_k$ is a component triangle in $T(A)$ if and only if the corresponding triangle $b_ib_jb_k$ is a component triangle in $T(B)$. 
If there exists a joint triangulation of $A$ and $B$, then $a_i a_j$ is an edge of the convex hull of $A$ if and only if the corresponding edge $b_i b_j$ is an edge of the convex hull of $B$. 
Proof

For the convex hull edge $a_1a_2$, there is only one triangle $a_1a_2a_6$ in $T(A)$ whereas for the corresponding edge $b_1b_2$, there are two triangles $b_1b_2b_5$ and $b_1b_2b_6$ in $T(B)$. 
A triangle $a_i a_j a_k$ is said to be an empty triangle in $A$ if it does not contain any point of $A$ in its interior.

Let $S_A$ denote the set of all empty triangles in $A$ whose corresponding triangles in $B$ are empty triangles in $B$. Let $S_B$ be the set of triangles corresponding to the triangles in $S_A$.

It follows from the definition of a joint triangulation that only triangles from $S_A$ and $S_B$ can be component triangles in a joint triangulation of $A$ and $B$. 
Let \( a_ia_ja_k \) and \( a_ia_ja_l \) be two component triangles in \( S_A \) lying on opposite sides of their common edge \( a_ia_j \). If component triangles \( b_ib_jb_k \) and \( b_ib_jb_l \) also lie on opposite sides of their common edge \( b_ib_j \), then \( a_ia_ja_l \) is called a successor triangle of \( a_ia_ja_k \) on the edge \( a_ia_j \) and vice versa. Analogously, \( b_ib_jb_l \) is also called a successor triangle of \( b_ib_jb_k \) on the edge \( b_ib_j \) and vice versa.
On the edge $a_6a_7$, $a_6a_7a_8$ and $a_6a_7a_2$ are successor triangles. The corresponding triangles $b_6b_7b_8$ and $b_6b_7b_2$ are also successor triangles on the edge $b_6b_7$.

Intuitively, if a triangle $ijk$ is a component triangle in a joint triangulation, one of the successors on each edge of $ijk$, that is not a convex hull edge, is also a component triangle in the joint triangulation.
Let $S$ denote the maximal subset of triangles in $S_A$ and $S_B$ such that each triangle $ijk$ in $S$ has at least one successor triangle in $S$, on the edges $ij$, $jk$ and $ki$ that are not convex hull edges. We call triangles in $S$ as *legal triangles* and $S$ is called the set of legal triangles.
Necessary Condition 2

If there exists a joint triangulation of $A$ and $B$, then the set of legal triangles $S$ is not empty.

**Conjecture:** There exists a joint triangulation of $A$ and $B$ if and only if $A$ and $B$ satisfy the two necessary conditions.
Testing necessary condition

- The first necessary condition can be tested by traversing the boundary of the convex hulls of A and B in $O(n)$ time after computing the convex hulls of A and B in $O(n \log n)$ time.

- For testing the second necessary condition, the algorithm starts by computing all empty triangles in A and B which can be at most $O(n^3)$ time. So, $S_A$ and $S_B$ can be computed in $O(n^3)$ time.

- For every non-convex hull edge $ij$ of all triangles in $S_A$ and $S_B$, the algorithm checks for successor triangles on $ij$. If $ij$ satisfies the successor condition, then there are successor triangles on the edge $ij$. Otherwise, all triangles in $S_A$ and $S_B$ with $ij$ as an edge are removed from $S_A$ and $S_B$.

- The process of checking successors and deleting triangles can be done in $O(n^3)$ time using a queue.

Theorem

Given two sets $A$ and $B$ of $n$ points in the plane, the two necessary conditions for a joint triangulation of $A$ and $B$ can be tested in $O(n^3)$ time.
Constructing joint triangulation

- Constructing a joint triangulation of $A$ and $B$ involves finding a subset $T$ of legal triangles in $S$ forming a triangulation in $A$ and the corresponding triangulation in $B$.
- The algorithm uses a greedy method to obtain $T$. Initialize $S' = S$ and $T = \emptyset$.
- Take any triangle $ijk$ from $S'$, add it to $T$ and delete all triangles in $S'$ that intersect the interior of the triangle $ijk$ in either $A$ or $B$.
- Repeat this process until $S'$ becomes empty.
- Experimentally, we have observed that the triangles in $T$ form a joint triangulation of $A$ and $B$.
- Our software for experimentation is available at (http://www.tcs.tifr.res.in/~ghosh/Joint-triangulation/joint-triangulation.html).
- However, we could not prove this claim.
A joint triangulation of two simple polygons $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$.

**Lemma:** All edges of the triangles in a joint triangulation of $A$ and $B$ must belong to visibility graphs of $A$ and $B$ respectively.
Let $IVG(A)$ denote the sub-graph of the visibility graph of $A$ such that $a_ia_j$ belongs to $IVG(A)$ if and only if $b_ib_j$ belongs to the visibility graph of $B$. Analogously, we define $IVG(B)$.

Let $SUB(A)$ denote the set of all sub-polygons of $A$ (including $A$ itself) that can be formed by cutting $A$ using only one diagonal of $IVG(A)$.

So, the size of sub-polygons in $SUB(A)$ varies from 3 to $n$.

Since all sub-polygons of three vertices in $SUB(A)$ (say, $Q_{1,3}, Q_{2,3}, \ldots$) admit joint triangulations as they are triangles, $M(Q_{1,3}), M(Q_{2,3}), \ldots$ are set to be true.


Then the procedure considers sub-polygons $Q_{1,4}$, $Q_{2,4}, \ldots$ of $\text{SUB}(A)$ having four vertices.

Let $Q_{1,4} = (a_i, a_{i+1}, a_{i+2}, a_{i+3})$. So, $a_ia_{i+3}$ is the diagonal of $\text{IVG}(A)$ used to cut $A$ to form $Q_{1,4}$.

Let $a_k$ be a vertex of $Q_{1,4}$ such that edges $a_ia_k$ and $a_ka_{i+3}$ belong to $\text{IVG}(A)$.

If no such $v_k$ exists, then set $M(Q_{1,4})$ to false.

If $a_{i+1} = a_k$ and the triangle $(a_{i+1}, a_{i+2}, a_{i+3})$ admits triangulation, then set $M(Q_{1,4})$ to true.

If $a_{i+2} = a_k$ and the triangle $(a_i, a_{i+1}, a_{i+2})$ admits triangulation, then set $M(Q_{1,4})$ to true.

Otherwise, set $M(Q_{1,4})$ to false.
Similarly, the procedure considers sub-polygons $Q_{1,5}, Q_{2,5}, \ldots$ of $SUB(A)$ having five vertices by locating all possible such vertices $a_k$.

This process is repeated till the sub-polygon of size $n$ (i.e., $A$) is considered.

**Theorem:** Given two simple polygons $A$ and $B$ of $n$ vertices, a joint triangulation of $A$ and $B$ can be constructed in $O(n^3)$ time.
Summery

- We establish two necessary conditions for a joint triangulation of two sets of $n$ points in the plane and conjecture that they are sufficient.
- We show that these necessary conditions can be tested in $O(n^3)$ time.
- For the problem of a joint triangulation of two simple polygons of $n$ vertices, we propose an $O(n^3)$ time algorithm for constructing a joint triangulation using dynamic programming.
Thank you.