Permutations

2.1 A permutation in $S_n$ is a transposition if it has one cycle with two elements and $n-2$ cycles with one element each.

(a) Show how you will write the cycle $(a_1, a_2 \ldots, a_m)$ as a product of $m-1$ transpositions.
(b) If $\sigma \in S_n$ is a permutation with $k$ cycles, show that $\sigma$ can be written as a product of $n-k$ transpositions.
(c) Let $\sigma \in S_n$ be a permutation with $k$ cycles.
(d) Consider the transposition $(1, 2)$. How many cycles can $(1, 2) \cdot \sigma$ have? Can it be written as a product of fewer than $n-k$ transpositions?

2.2 Suppose you are given an array $(A[i] : i = 1, 2, \ldots, n)$, which contains the numbers $1, 2, \ldots, n$ stored in some order. To move the elements of the array, the only operation we are allowed is swap$(i, j)$, where $i$ and $j$ are distinct indices in the range $1, 2, \ldots, n$. This operation exchanges the values in $A[i]$ and $A[j]$. Give a linear-time algorithm that uses the swap operation repeatedly so that in the end the element $i$ is in the location $A[i]$. Your program can use an auxiliary bit-array $(B[i] : i = 1, 2, \ldots, n)$, and a constant number of other variables, each holding an integer in the range $0, 1, \ldots, n+1$. How many swaps will your algorithm need if the initial content of the array corresponds to a permutation with $k$ cycles (that is, if we define the permutation $\sigma : [n] \to [n]$ by $\sigma[i] \triangleq A[i]$, then $\sigma$ has $k$ cycles)? Can an algorithm (not necessarily linear-time) use even fewer swaps?

2.3 Consider a permutation $\rho \in S_n$ with exactly one non-trivial cycle $(a_1, a_2, \ldots, a_m)$. Suppose $\sigma \in S_n$. Describe the cycles of the permutation $\sigma \cdot \rho \cdot \sigma^{-1}$.

Inclusion-exclusion

2.5 A surjection is an onto function i.e. every element of the co-domain has a pre-image. Show that the number of surjections from $[s]$ to $[n]$ is

$$\sum_{k=0}^{s} (-1)^k \binom{n}{k} (n-k)^s.$$ 

Hence, conclude that the above expression is 0 iff $s < n$.

2.6 (Bonferroni’s inequalities.) Let $A_1, A_2, \ldots, A_k \subseteq [n]$. For $S \subseteq [k]$, let $A_S \triangleq \cap_{s \in S} A_s$, $A_\emptyset \equiv [n]$. Then, for $0 \leq r \leq k$, show that

$$|A_1 \cup A_2 \cup \ldots \cup A_k| \geq \sum_{S \subseteq [k] : |S| \leq r} (-1)^{|S|}|A_S|, \quad r \text{ odd};$$
$$|A_1 \cup A_2 \cup \ldots \cup A_k| \leq \sum_{S \subseteq [k] : |S| \leq r} (-1)^{|S|}|A_S|, \quad r \text{ even}.$$ 

That is, show that the successive steps in the inclusion-exclusion formula alternately bound the final value from above and below.
2.7 (a) (Möbius inversion.) Let \( f, g : \{1, 2, \ldots, n\} \to \mathbb{C} \) be two functions. Suppose
\[
f(n) = \sum_{d | n} g(d).
\]
Let \( \mu(\cdot) \) be the Möbius function on positive integers defined as follows: Let positive integer \( x \) have the prime factorisation \( x = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \). Then,
\[
\mu(x) = \begin{cases} 
0 & \text{if } r_i \geq 2 \text{ for some } 1 \leq i \leq k \\
(-1)^k & \text{otherwise}.
\end{cases}
\]
Show that
\[
g(n) = \sum_{d | n} \mu(n/d) f(d).
\]
(b) Prove the following identity via a counting argument:
\[
n = \sum_{d | n} \varphi(d).
\]
Hence, derive a formula for \( \varphi(\cdot) \) in terms of \( \mu(\cdot) \).

2.8 (a) Consider the \( 2^n \times 2^n \) matrix \( I \) with rows and columns indexed by the subsets of \( [n] \) defined as follows:
\[
I_{A,B} = \begin{cases} 
1 & \text{if } A \subseteq B \\
0 & \text{otherwise}.
\end{cases}
\]
The matrix \( I \) is known as the set inclusion matrix. Find \( I^{-1} \) explicitly i.e. you should be able to write down \( I_{A,B}^{-1} \) for any \( A, B \subseteq [n] \).
(b) Show that the expression for \( I^{-1} \) derived above gives rise to the general inclusion-exclusion formula.
(c) Using the first part of this exercise or otherwise, show that the set disjointness matrix \( D \) defined as:
\[
D_{A,B} = \begin{cases} 
1 & \text{if } A \cap B = \{\} \\
0 & \text{otherwise},
\end{cases}
\]
is invertible. Find an explicit expression for \( D^{-1} \).

2.9 Read and understand the solution to the following problem, and submit solutions for the remaining. Graph embeddings: Our graphs are undirected and simple with vertex set \( [n] \). Let \( G_1 \) and \( G_2 \) be graphs with \( m \) edges. For graphs \( H \) and \( G \), we say that \( f : [n] \to [n] \) is an embedding of \( H \) in \( G \) if (a) \( f \) is one-to-one and onto, and (b) for all \( \{i, j\} \in E(H) \), we have \( \{f(i), f(j)\} \in E(G) \). Suppose for each graph \( H \) with \( m-1 \) edges, the number of subgraphs of \( G_1 \) that are isomorphic to \( H \) is equal to the number of subgraphs of \( G_2 \) that are isomorphic to \( H \). Then, show that for every graph \( H \) with at most \( m-1 \) edges, the number of embeddings of \( H \) in \( G_1 \) is equal to the number of embeddings of \( H \) in \( G_2 \).
Solution: Order the edges of $G_1$ as $e_1,e_2,\ldots,e_m$ and $G_2$ as $f_1,f_2,\ldots,f_m$ so that the graph $G_1-e_i$ is isomorphic to $G_2-f_i$; fix an embedding $\sigma_i$ of $G_1-e_i$ in $G_2-f_i$ for each $i \in [m]$. Let $H$ be some graph on $[n]$ with $k \leq m-1$ edges. We want to show that the number of embeddings of $H$ in $G_1$ is equal to the number of embeddings of $H$ in $G_2$. Let

$$S_1 \triangleq \{(f,i) : f \text{ is an embedding of } H \text{ in } G_1-e_i\};$$

and

$$S_2 \triangleq \{(f,i) : f \text{ is an embedding of } H \text{ in } G_2-f_i\}.$$  

Note that $(f,i) \in S_1$ if and only if $(\sigma_i \cdot f,i) \in S_2$; so, $|S_1| = |S_2|$. Now, if $f$ is an embedding of $H$ in $G$, there are exactly $m-k$ indices $i$ such that $f$ is an embedding of $H$ in $G-e_i$. It follows that

the number of embeddings of $H$ in $G_1 = \frac{1}{m-k}|S_1|$.  

Similarly,

the number of embeddings of $H$ in $G_2 = \frac{1}{m-k}|S_2|$.  

But, we just argued that $|S_1| = |S_2|$. So, the number of embeddings of $H$ in $G_1$ is equal to the number of embeddings of $H$ in $G_2$.

2.10 Let $N$ be a finite set, and let $\mathcal{P}(N)$ be the power set of $N$. Let $f : \mathcal{P}(N) \to \mathbb{R}$. Define $e : \mathcal{P}(N) \to \mathbb{R}$ by

$$e(T) \triangleq \sum_{S : S \supseteq T} f(S). \tag{1}$$

Suppose for some subset $T$ of size $m$ we have $e(T) \neq 0$, but $e(T') = 0$ for all proper subsets $T'$ of $T$. Show that there are at least $2^m$ sets $S \subseteq N$ such that $f(S) \neq 0$.

2.11 Let $N = \binom{[n]}{2}$. Let $G$ be a graph on $[n]$ and $m$ edges, that is, $G \in \binom{N}{m}$. Let the function $f : \mathcal{P}(N) \to \mathbb{R}$ be defined as follows: if $G'$ has exactly $m$ edges, then $f(G')$ is the number of embeddings of $G$ in $G'$; if $G'$ does not have exactly $m$ edges, then $f(G') \triangleq 0$. Using this $f$, define $e : \mathcal{P}(N) \to \mathbb{R}$ as in (1). Show that $e(H)$ is exactly the number of embeddings of $H$ in $G$.

2.12 Observe that the number of $G'$ for which $f(G') \neq 0$ is at most $n!$ (Why?). Use [2.9] and [2.10] to conclude that if two graphs $G_1$ and $G_2$ with $m$ edges have the same list (or deck) of subgraphs with $m-1$ edges, then the resulting $e$’s obtained from them (as in [2.10]) take the same value for all graphs $H$ with at most $m-1$ edges. Conclude that if $G_1$ and $G_2$ are not isomorphic, then $2 \cdot n! \geq 2^m$. That is, if $m > 1 + \log_2(n!)$, then the graph can be reconstructed from its deck. Note that Lovász’s proof, presented in class, showed that we can reconstruct the graph provided it has more than $\frac{1}{2}\binom{n}{2}$ edges.

Please send me email (jaikumar@tifr.res.in) when you spot errors. – Jaikumar