Homewok 3 (5 problems, due: 3 November 2008)

C3.1 Let $G = (V_1, V_2, E)$ be a bipartitite graph. Suppose there is a non-negative integer d such that for all $S \subseteq V_1$ we have $|N(S)| \geq |S| - d$. Show that G has a matching of size at least $|V_1| - d$.

C3.2 For a graph G, a vertex cover is a set of vertices of G such that every edge has at least one end point in the set. Show that for a bipartite graph the size of the largest matching and the size of the smallest vertex cover are equal. What about non-bipartite graphs?

C3.3 In a graph, the degree of a vertex is the number of edges incident on it. A graph is r-regular if every vertex has degree exactly r. Show that every r-regular bipartite graph is a union of r perfect matchings. Conclude that any bipartite graph with maximum degree r is the union of at most r matchings. Observe that the theorem holds even if multiple edges are allowed.

C3.4 Consider the complete bipartite graph $K_{n,n}$ ($|V_1|, |V_2| = n, E = V_1 \times V_2$). Let $w : E \rightarrow \mathbb{R}$. Let S_w denote the set of all perfext matchings of $K_{n,n}$ with maximum weight (with respect to the given function w). Show the following theorem of Sundar Vishwanathan:

for every $w : E \rightarrow \mathbb{R}$ *there is a* $w' : E \rightarrow \{0, 1\}$ *such that* $S_w = S_{w'}$.

Hint: Use C3.4.

C3.5 Let M be an $n \times n$ with positive real entries. M is said to be doubly stochastic if the sum of each row and each column is 1. Show that every doubly stochastic matrix is a convex combination of permutation matrices, that is

$$M = \sum_{\pi} w_\pi M_\pi,$$

where w_π are non-negative real weights that add up to 1, and M_π is the permutation matrix corresponding to π. (This is called the Birkhoff-von Neumann theorem.)

Please send me email (jaikumar@tifr.res.in) when you spot errors. – Jaikumar