Better Lower Bounds for Monotone Threshold Formulas

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Abstract

We show that every monotone formula that computes the threshold function $\mathrm{TH}_{k,n}$, $2 \le k \le \frac{n}{2}$, has size at least $\left\lfloor \frac{k}{2} \right\rfloor n \log(\frac{n}{k-1})$. The same lower bound is shown to hold in the stronger monotone directed contact networks model.

1 Introduction

A *formula* is a Boolean circuit whose underlying graph is a tree. A formula with n variables computes a Boolean function from $\{0,1\}^n$ to $\{0,1\}$ in a natural way. The size of a formula is the number of occurrences of variables in it, that is, the number of leaves in its underlying tree. A *monotone formula* is a formula over the basis {AND,OR}, that is, a formula each of whose gates is an AND or an OR.

The threshold function $\mathrm{TH}_{k,n}$ is a Boolean function that takes the value 1 precisely when at least k of its n variables are assigned 1. Threshold functions play a central role in the investigation of the computational complexity of Boolean functions (see Boppana and Sipser [3], Wegener [28]). Their complexity has been studied in various circuit models. In this paper, we show lower bounds on the size of monotone formulas computing threshold functions.

We show that every monotone formula computing $\mathrm{TH}_{k,n}$, $2 \le k \le \frac{n}{2}$, has size at least $\left\lfloor \frac{k}{2} \right\rfloor n \log(\frac{n}{k-1})$. In the monotone formulas model, the complexities of computing $\mathrm{TH}_{k,n}$ and $\mathrm{TH}_{n-k+1,n}$ are the same. Hence, the lower bound of $\left\lfloor \frac{k}{2} \right\rfloor n \log(\frac{n}{k-1})$ holds for the function $\mathrm{TH}_{n-k+1,n}$, $2 \le k \le \frac{n}{2}$, as well.

We obtain our lower bound for monotone formulas by showing that every *monotone directed contact network* (see Definition 2.1) computing $\mathrm{TH}_{k,n}$, $2 \le k \le \frac{n}{2}$, has size at least $\left\lfloor \frac{k}{2} \right\rfloor n \log(\frac{n}{k-1})$. Since every monotone formula can be converted to a monotone directed contact network of the same size, the lower bounds for monotone formulas follow from the lower bounds for monotone directed contact networks.

1.1 Related work

The computation of threshold functions by formulas has been widely studied. Over the complete binary basis, Paterson, Pippenger, and Zwick [18] showed that all threshold functions can be

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computed by formulas of size $O(n^{3.13})$. For this basis, Pudlák [20] showed a lower bound of $\Omega(n \log\log n)$ for computing $\mathrm{TH}_{k,n}$, $2 \le k \le \frac{n}{2}$; Fischer, Meyer, and Paterson [6] showed a lower bound of $\Omega(n \log n)$ for the majority function $\mathrm{TH}_{\lfloor n/2 \rfloor, n}$.

Over the basis {AND,OR,NOT}, Paterson, Pippenger, and Zwick [18] showed that $\mathrm{TH}_{k,n}$ can be computed by formulas of size $O(n^{4.57})$. Lower bounds on the size of such formulas were shown by Hansel [8], Krichevskii [13], and Khrapchenko [12]. Hansel and Krichevskii showed a lower bound of $\Omega(n \log n)$ for computing $\mathrm{TH}_{2,n}$. This implies an $\Omega(n \log n)$ lower bound for all threshold functions $\mathrm{TH}_{k,n}$, $2 \le k \le n-1$. Khrapchenko showed that any such formula computing $\mathrm{TH}_{k,n}$ has size at least $k(n-k+1)$.

The existence of polynomial size *monotone* formulas for computing $\mathrm{TH}_{k,n}$ is implied by the $O(\log n)$ depth sorting network due to Ajtai, Komlós, and Szemerédi [1]. The existence of more efficient monotone threshold formulas was shown by Valiant [27] and Boppana [2]. Valiant showed that the majority function ($TH_{\lfloor n/2 \rfloor, n}$) can be computed using monotone formulas of size $O(n^{5.3})$. Boppana generalized Valiant's result and showed that $\mathrm{TH}_{k,n}$ can be computed by monotone formulas of size $O(k^{4.3} n \log n)$. The lower bounds due to Hansel, Krichevskii, and Khrapchenko, stated above, hold for monotone formulas as well. Before this work, these were the best lower bounds known for monotone formulas. The result of Hansel and Krichevskii was generalized by Snir [26] to obtain an $\Omega(kn \log(\frac{n}{k-1}))$ lower bound in the context of hypergraph covering. Snir's result implies an $\Omega(kn \log(\frac{n}{k-1}))$ lower bound on the size of certain restricted depth three formulas computing $\mathrm{TH}_{k,n}$ (see [17, 23]). However, it is not clear how Snir's result may be used to derive our results for monotone threshold formulas.

Related to the monotone formulas model is the model of the *monotone contact networks.* Several variants of this model have been studied in the past (see Razborov [22]). The most powerful among these are the *monotone contact-rectifier networks.* Markov [16] showed that the size of the smallest such network for computing $\mathrm{TH}_{k,n}$ is precisely $k(n-k+1)$. For general contact-rectifier networks (where negations are permitted), Lupanov [15] showed an upper bound of $O(n^{3/2})$ on the complexity of computing any threshold function; Razborov [21] showed a lower bound of $\Omega(n \log\log\log^* n)$ for the majority function.

Another variant is the model of the *monotone directed contact networks.* In this model, the underlying graph is directed (see Definition 2.1), and all the labels are variables [2]. If constant 1's are allowed to appear as labels, then these networks reduce to the contact-rectifier networks discussed above. In this paper, we show a lower bound of $\left\lfloor \frac{k}{2} \right\rfloor n \log(\frac{n}{k-1})$ on the size of any monotone directed contact network (without 1's) that computes $\mathrm{TH}_{k,n}$, $2 \le k \le \frac{n}{2}$. An upper bound of $(k-1)(n-k+2)\lceil \log(n-k+2) \rceil$ for computing $\mathrm{TH}_{k,n}$, $2 \le k \le n-1$, has been shown by Radhakrishnan and Subrahmanyam [24]. Thus, our results are close to optimal for small values of k.

The most widely studied monotone contact networks are the monotone undirected contact networks [25]. For such networks, the underlying graph is undirected. Note that the presence of constant 1's as labels is inconsequential in this case because edges with such labels can be eliminated by collapsing them and identifying their end points. For this model, the results of Dubiner and Zwick [4] imply an upper bound of $O(k^{3.99} n \log n)$ for computing $\mathrm{TH}_{k,n}$ and $\mathrm{TH}_{n-k+1,n}$. The results of Markov [16] for contact-rectifier networks imply a lower bound of $k(n-k+1)$. Krichevskii [13] showed a lower bound of $\Omega(n \log n)$ for $\mathrm{TH}_{2,n}$. It is easy to see that an undirected contact network can be converted to a directed contact network by replacing each undirected edge by two directed edges. Hence, our lower bound for monotone directed contact networks can be translated to this model, losing at most a factor of two in the translation.

The relation between communication complexity and formula complexity was exploited by Karchmer and Wigderson [11] to show very strong lower bounds for computing the *st-connectivity* function using monotone formulas. However, as noted in [10, page 60], the communication complexity approach does not seem to shed much light on the computation of threshold functions. To show lower bounds for computing $\mathrm{TH}_{k,n}$ using monotone formulas and monotone contact networks, we make use of a result due to Fredman and Komlós [5] on graph covering.

1.2 Overview

The rest of this paper is organized as follows. In Section 2, we introduce the notation. In Section 3, we recall the result of Fredman and Komlós on graph covering. The lower bounds for monotone formulas and monotone directed contact networks are shown in Section 4.

2 Notation

Suppose f is a Boolean function with n variables $x_1, x_2, \ldots, x_n$. We say that f *accepts* $T \subseteq \{x_1, x_2, \ldots, x_n\}$ if f evaluates to 1 when all the variables in T are given the value 1 and the remaining variables are given the value 0. We say that f is *l-immune* if it accepts no T with $|T| \leq l$. Thus, the threshold function $\mathrm{TH}_{k,n}$ is $(k-1)$-immune.

While referring to graphs, we shall use the following terminology. The size of the largest independent set in a graph G will be denoted by $\alpha(G)$; $\mathrm{size}(G)$ will denote the number of non-isolated vertices in G. A function f with domain $V(G)$ will be called a *coloring* of G if $f(i) \neq f(j)$ whenever $(i,j) \in E(G)$. For a graph G, G^N will denote the subgraph of G induced by the non-isolated vertices of G. For two graphs F and G on the same set of vertices V, we denote their union $(V, E(F) \cup E(G))$ by $F \cup G$.

Definition 2.1 *A* monotone directed contact network *is a directed graph with two distinguished vertices s and t. Each edge of the graph has a variable as its label. We use $vars(N)$ to denote the variables of the contact network N. For a pair (v,w) of vertices, the contact network computes the Boolean function $f_{v,w}$ as follows. On an assignment $y : vars(N) \to \{0,1\}$, the label on each edge is set to 0 or 1 in accordance with y. Then $f_{v,w}(y) = 1$ if there is a path from v to w using only the edges with label 1, and $f_{v,w}(y) = 0$ otherwise. We refer to the function $f_{s,t}$ as the function computed by N and denote it by f_N. The* size *of a network is the number of edges in it.*

In a *monotone contact-rectifier network*, the constant 1 is also allowed to appear as a label. The size of a monotone contact-rectifier network is the number of edges that have *variables* as labels. In a *monotone undirected contact network*, the underlying graph is undirected. From now on, when we say *monotone contact network*, or just *contact network*, we shall mean monotone *directed* contact network.

We say that a contact network N *accepts* a set A if the function f_N accepts A. We say that N is r-*immune* if f_N is r-immune. We shall extend this terminology and apply it to the vertices of the contact network. For example, we shall say that a vertex p of the contact network N accepts a set A if the function $f_{p,t}$ accepts A. Thus, the contact network accepts precisely those inputs that are accepted by the distinguished vertex s. Similarly, we say that the vertex p is r-immune if $f_{p,t}$ is r-immune.

Definition 2.2 A *depth two contact network* is a contact network where each edge is incident on s or t. Further, s has indegree zero, and t has outdegree zero.

3 Graph covering

We shall need the following standard definition from information theory.

For a random variable X with finite support, its entropy is given by

$$H(X) = -\sum_x \Pr[X = x] \log \Pr[X = x].$$

The entropy of a function f will be the entropy of the random variable $f(X)$, where X assumes values in the domain of f with uniform distribution.

The following information theoretic measure on graphs was introduced by Fredman and Komlós [5].

Definition 3.1 (Coloring Entropy) Let G be a graph. Let f be the coloring of the graph G^N with minimum entropy. The coloring entropy of G is given by

$$H(G) = \frac{\text{size}(G)}{|V(G)|} H(f).$$

(If $E(G)$ is empty, then $H(G) = 0$.)

The following lemma is due to Fredman and Komlós [5].

Lemma 3.2 Let $G, G_1, G_2, \ldots, G_l$ be graphs on the same set of vertices. Let $G = G_1 \cup G_2 \cup \ldots \cup G_l$. Then

$$\sum_{i=1}^{l} H(G_i) \geq \log(\frac{|V(G)|}{\alpha(G)}).$$

∎

Since every bipartite graph has a coloring with entropy at most 1, we have the following corollary to Lemma 3.2.

Corollary 3.3 Let $G_1, G_2, \ldots, G_l$ be bipartite graphs on the same set of vertices. Let $G = G_1 \cup G_2 \cup \ldots \cup G_l$. Then

$$\sum_{i=1}^{l} \text{size}(G_i) \geq |V(G)| \log(\frac{|V(G)|}{\alpha(G)}).$$

∎

4 Monotone formulas

In this section, we shall extend the results of Hansel, Krichevskii, and Khrapchenko and show better lower bounds on the size of monotone formulas computing $\text{TH}_{k,n}$.

Note that a monotone formula can be converted to a monotone contact network of the same size by representing the OR's in parallel and the AND's in series. Hence, to show lower bounds on the size of monotone formulas computing $\text{TH}_{k,n}$, it suffices to show lower bounds on the size of contact networks computing $\text{TH}_{k,n}$.

The following lemma is implicit in the work of Krichevskii [13].

Lemma 4.1 *Every* 1*-immune monotone contact network* N *can be converted to a depth two contact network* $\hat{N}$ *such that:*

1. $\hat{N}$ *is* 1*-immune;*
2. *Size of* $\hat{N}$ *is at most the size of* N*;*
3. *Every input accepted by* N *is accepted by* $\hat{N}$*.*

Proof. We shall first convert N to a network N' of the same size and accepting the same inputs as N. We shall ensure that in the network N', if the vertex v has an edge from s with label x_i, then no edge leaving vertex v has label x_i. This property will help us to obtain the desired depth two network $\hat{N}$.

Suppose $\text{vars}(N) = \{x_1, x_2, \ldots, x_n\}$. Let V_1 be the set of vertices in N that are reachable from s using only those edges that have label x_1. Delete all edges incident on vertices in V_1 with label x_1. Add new edges connecting s to each vertex in $V_1 - \{s\}$. Label the new edges with x_1. Repeat this procedure for the other labels $x_2, x_3, \ldots, x_n$. The final network thus obtained is N'. In each phase, the number of new edges added is at most the number of edges deleted. Hence, $\text{size}(N') \leq \text{size}(N)$. Also, after each phase, the new network accepts exactly the same inputs as the old network. Hence, $f_N = f_{N'}$. In particular, since N is 1-immune, N' is 1-immune. Further, N' has the property stated above.

Let v be an internal vertex of N'. Let A_v be the set of labels on the edges (s, v) and B_v be the labels on the edges (v, w) leaving v. Then A_v and B_v are disjoint sets, and the size of N' is at least $\sum_v |A_v| + |B_v|$.

The network $\hat{N}$ is constructed as follows. The set of vertices for $\hat{N}$ is the same as the set of vertices for N'. For each internal vertex v, add $|A_v|$ edges of the form (s, v), one for each label in A_v. Similarly, add $|B_v|$ edges of the form (v, t), one for each label in B_v.

Clearly, the size of $\hat{N}$ is at most the size of N'. Since A_v is disjoint from B_v, $\hat{N}$ is 1-immune. It only remains to verify that $\hat{N}$ accepts all the inputs that N' accepts. Suppose y is accepted by N'. Then, on input y, there is a path from s to t all of whose labels are 1. Since N' is 1-immune, this path must have length at least two. Let v_1 be the second vertex on this path. Then the edge (s, v_1) and an edge leaving v_1 are set to 1 on input y. Thus, there is a path generated from s to t via v_1 in $\hat{N}$. Hence, $\hat{N}$ accepts y. ∎

Lemma 4.2 *Let N be a 1-immune monotone contact network that accepts all the sets of size k. Then the size of N is at least $n \log(\frac{n}{k-1})$.*

Proof. By Lemma 4.1, we may assume that N is a depth two contact network. For each internal vertex v of N, let A_v be the set of labels that appear on the edges of the form (s, v) and B_v be the set of labels that appear on the edges of the form (v, t). Let G_v be the undirected bipartite graph with vertex set $V(G_v) = \{x_1, \ldots, x_n\}$ and edge set

$$E(G_v) = \{\{x_i, x_j\} : \ x_i \in A_v \text{ and } x_j \in B_v\}.$$

Note that the size of N is at least $\sum_v \mathrm{size}(G_v)$. Let $G = \bigcup_v G_v$.

Suppose N accepts the set $I \subseteq \mathrm{vars}(N)$. We shall show that I is not an independent set of G. Since N accepts I, on the input corresponding to I, there is a path from s to t all of whose labels are 1. That is, for some internal vertex v in N, $A_v \cap I \neq \emptyset$ and $B_v \cap I \neq \emptyset$. It follows that I is not an independent set in G_v. Hence, I is not an independent set in G. Since N accepts all sets of size k, G has no independent set of size k. It follows that $\alpha(G) \leq k - 1$. From Corollary 3.3, we conclude that that $\sum_v \mathrm{size}(G_v) \geq n \log(\frac{n}{k-1})$. The lemma follows from this. ∎

Theorem 4.3 Let $k \geq 2$, and let N be a monotone contact network computing T_k^n. Then

$$\mathrm{size}(N) \geq \left\lfloor \frac{k}{2} \right\rfloor n \log(\frac{n}{k-1}).$$

Proof. We shall use induction on d to show that the following assertion holds for all positive integers d.

If N is a $(2d-1)$-immune monotone contact network that accepts all sets of size k, then $\mathrm{size}(N) \geq dn \log(\frac{n}{k-1})$.

The basis case, when $d = 1$, is Lemma 4.2 above. Assume that the assertion is true with $d = r$, for some positive integer r. We shall show that the assertion holds for $d = r + 1$.

Suppose that N is a $(2(r+1)-1)$-immune contact network that accepts all sets of size k. Let V' be the set of vertices that accept some input of size at most two. Note that t is in V' and s is not in V'. Let V_2 be those vertices in V' that are 1-immune. Let $V_1 = V' - V_2$.

Let L be the network obtained from N by collapsing all the vertices in V' to form the new sink t'. The source of L will be s. Let M be the network obtained from N by deleting all the vertices

Figure 1: The induction step

outside V' and the edges incident on them. The source s' of M is obtained by collapsing all the vertices in V_2. The sink of M will be t. (See Figure 1.) Note that $\text{size}(N) \geq \text{size}(L) + \text{size}(M)$.

Claim 1: L is $(2r-1)$-immune, and L accepts all sets of size k.

Proof: Suppose L accepts a set A. We shall show that $|A| \geq 2r$. It will follow that L is $(2r-1)$-immune. Since L accepts A, there is a vertex v in V' such that $f_{s,v}$ accepts A. By the definition of V', we have that $f_{v,t}$ accepts a set B of size at most two. Then N accepts $A \cup B$. Since N is $(2(r+1)-1)$-immune,

$$|A| \geq 2(r+1) - |B| \geq 2r.$$

Since $t \in V'$, L accepts all inputs that N accepts. Since N accepts all sets of size k, L accepts all sets of size k. (End of Claim 1.)

Claim 2: M is 1-immune, and M accepts all sets of size k.

Proof: Since every vertex in V_2 is 1-immune, it follows that M is 1-immune. Next, we show that M accepts all inputs that N accepts. Since N accepts all sets of size k, it will follow that M accepts all sets of size k. Let y be an input accepted by N. Then, on input y, there is a path p from s to t in N, all of whose labels are 1. Let v be the last vertex on this path that is 1-immune. (Since s is 1-immune, there is at least one such vertex.) All the vertices after v are not 1-immune; hence those vertices are in V_1. We claim that $v \in V_2$. Since the successor of v on the path is in V_1, v accepts a set of size at most two. Hence, v is a 1-immune vertex in V'. It follows that v is in V_2. Consider the part of the path p from v to t. This is contained entirely in M. Hence, M accepts y. (End of Claim 2.)

From Claim 1 and the assertion with $d = r$, we obtain

$$\mathrm{size}(L) \geq rn\log(\frac{n}{k-1}). \tag{1}$$

From Claim 2 and Lemma 4.2, we obtain

$$\mathrm{size}(M) \geq n\log(\frac{n}{k-1}). \tag{2}$$

Combining (1) and (2), we have

$$\mathrm{size}(N) \geq \mathrm{size}(L) + \mathrm{size}(M) \geq (r+1)n\log(\frac{n}{k-1}).$$

This completes the induction step.

We may now complete the proof of the theorem by taking $d = \left\lfloor \frac{k}{2} \right\rfloor$ in the above assertion. ∎

Corollary 4.4 Every monotone formula computing $\mathrm{TH}_{k,n}$, for $2 \leq k \leq \frac{n}{2}$, has size at least $\left\lfloor \frac{k}{2} \right\rfloor n\log(\frac{n}{k-1})$. ∎

5 Concluding remarks

Khrapchenko showed that any formula over the basis {AND,OR,NOT} computing $\mathrm{TH}_{k,n}$ has size at least $k(n-k+1)$. This is maximum for $k = (n+1)/2$, where it gives a lower bound of $0.25(n+1)^2$. For $k = \lfloor n/e \rfloor$, Corollary 4.4 gives a lower bound of $0.265n^2$ on the size of any *monotone* formula computing $\mathrm{TH}_{k,n}$. It will be of immense importance to show a lower bound of $\omega(n^2)$ on the size of formulas computing majority, even in the monotone case.

Our proof makes use of the monotonicity of the formula. Is there an $\Omega(kn\log\left(\frac{n}{k-1}\right))$ lower bound for computing $\mathrm{TH}_{k,n}$, $2 \leq k \leq \frac{n}{2}$, even when negations are allowed?

In the monotone formulas model, $\mathrm{TH}_{k,n}$ and $\mathrm{TH}_{n-k+1,n}$ have the same complexity. However, this is not true for monotone *directed* contact networks. While there is an $n\log n$ lower bound for $\mathrm{TH}_{2,n}$, there do exist linear size monotone directed contact networks computing $\mathrm{TH}_{n-1,n}$. The upper bound of $(k-1)(n-k+2)\lceil\log(n-k+2)\rceil$ for $\mathrm{TH}_{k,n}$, $2 \leq k \leq n-1$, shown in [24] relies on the networks being directed. For monotone undirected contact networks, the best upper bound known for computing $\mathrm{TH}_{n-1,n}$ is $O(n\log n)$. It has been shown in [7] that every monotone *undirected* contact network computing $\mathrm{TH}_{n-1,n}$ has size $\Omega(n\log\log\log n)$. Thus, unlike monotone directed contact networks, monotone undirected contact networks cannot compute $\mathrm{TH}_{n-1,n}$ in linear size. Is there a lower bound of $\Omega(kn\log\left(\frac{n}{k-1}\right))$ on the size of monotone undirected contact networks computing $\mathrm{TH}_{n-k+1,n}$, $2 \leq k \leq \frac{n}{2}$?

For certain planar undirected contact networks, a strong duality theorem holds (see [9, page 87]). This implies that for such planar undirected monotone contact networks, the complexities of computing $\mathrm{TH}_{k,n}$ and $\mathrm{TH}_{n-k+1,n}$ are the same.

Acknowledgments. I thank Endre Szemerédi for his generous contribution to this work. I am grateful to Ravi Boppana, Shiva Chaudhuri, Rafi Heiman, Zoli Király, Ilan Newman, K. V.

Subrahmanyam, Gábor Tardos, and Avi Wigderson for their comments on the earlier versions of this paper. I thank the referee for her comments and suggestions. Mark Kayll painstakingly corrected the numerous errors in the previous version of this paper; I thank him for his help.

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