Welcome to

CSS.328.1: Linear and Semidefinite Programming

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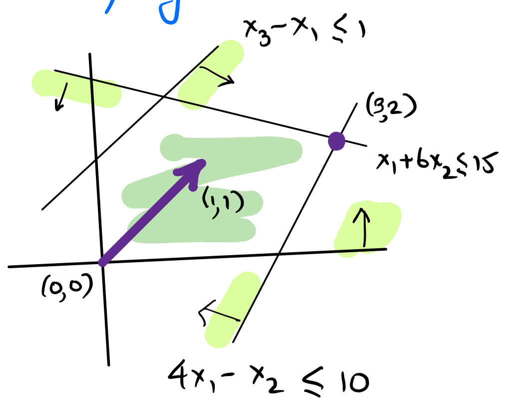
Tuesdays and Thursdays: 9:30 am to 11:00 am

Course grade: Homework (50%), Presentation (20%), Exam (30%)

A linear programming problem, linear program

$$\max_{(x_1,x_2)\in\mathbb{R}^2} \quad x_1 + x_2$$

Subject to
$$x_2 - x_1 \le 1$$
$$x_1 + 6x_2 \le 15$$
$$4x_1 - x_2 \le 10$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



Terminology

Objective function

Constraints

Non-negativity constraints

Feasible solution

Feasible region

Optimal solution

An LP may be infeasible, unbounded, or may have one or multiple optimal solutions.

Theorem: No other situation can occur.

An LP: $\displaystyle\max_{x \in \mathbb{R}^n} c^T x$ subject to $Ax \leq b$.

$c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

- A linear program is efficiently solvable, both in theory and practice.

Efficiently $\equiv$ in time that grows polynomially in the length of the input.

- Comparison with solving a system of linear equations.

efficient

not obvious!

Linear Equations	Gaussian elimination	Affine subspace
Linear Programs	Simplex method	Convex polyhedron

not efficient?

Examples

1. The diet problem

Requirement

Nutrient ↓ \ Dish →	D_1	D_2	D_3	...	R
Vitamin A	a_{11}	a_{12}	a_{13}	...	b_1
Calcium	a_{21}	a_{22}	a_{23}	...	b_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$
Fat	a_{m1}	a_{m2}	a_{m3}	...	b_m

Cost	c_1	c_2		

$$\min \quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

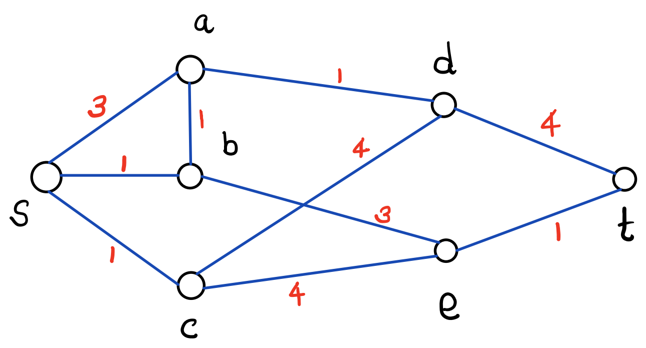
Subject to

$$A x \geq b$$

$$x \geq 0$$

Component wise

2. Flow in a network



What is the maximum rate of flow from s to t, if all flows respect capacity?

Direct the edges arbitrarily, and assign a variable for each (directed) edge.

$$x_{sa}, x_{sb}, x_{sc}, \dots, x_{d,t}, x_{et}.$$

The values assigned to these variables correspond to flows they carry. A negative value corresponds to flow in the opposite direction.

- For each vertex v other than s and t write a constraint

$$\boxed{\sum_{f \text{ enters } v} x_f - \sum_{f \text{ leaves } v} x_f = 0}$$

- For each edge f write

$$\boxed{-\mathrm{cap}(f) \le x_f \le \mathrm{cap}(f)}$$

- Maximize $\boxed{x_{sa} + x_{sb} + x_{sc}}$

$$
\begin{array}{c|c}
 & \begin{array}{ccc} (a,b) & \cdots & (c,e) \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} &
\begin{pmatrix}
+1 & & \\
-1 & & \\
 & & -1 \\
 & & \\
 & & +1
\end{pmatrix}
\end{array}
\begin{pmatrix} x_{ab} \\ \vdots \\ x_{ce} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

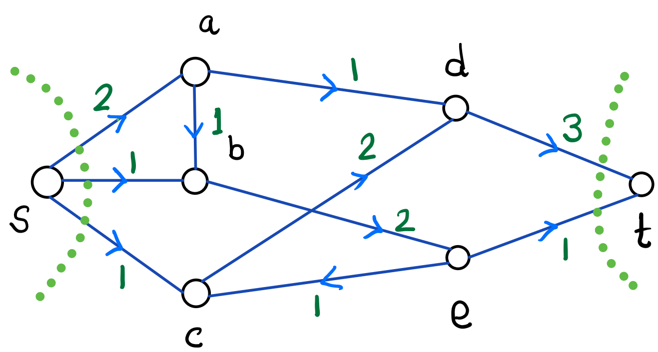
incidence matrix

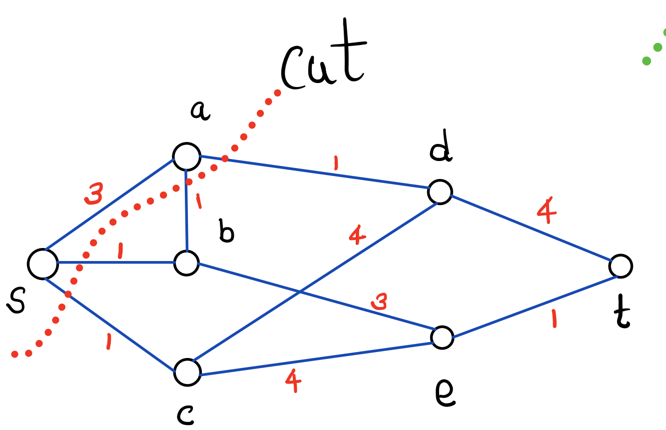
equality constraints

$$
\begin{array}{c|c}
 & \begin{array}{ccc} ab & \cdots & ce \end{array} \\
\begin{array}{c} ab \\ \vdots \\ ce \end{array} &
\begin{pmatrix}
1 & & \\
 & 1 & \\
 & & \ddots \\
 & & & 1
\end{pmatrix}
\end{array}
\begin{pmatrix} x_{ab} \\ \vdots \\ x_{ce} \end{pmatrix}
\leq
\begin{pmatrix} | \\ cap \\ | \end{pmatrix}
$$

$$
\begin{array}{c|c}
 & \begin{array}{ccc} ab & \cdots & ce \end{array} \\
\begin{array}{c} ab \\ \vdots \\ ce \end{array} &
\begin{pmatrix}
-1 & & \\
 & -1 & \\
 & & \ddots \\
 & & & -1
\end{pmatrix}
\end{array}
\begin{pmatrix} x_{ab} \\ \vdots \\ x_{ce} \end{pmatrix}
\leq
\begin{pmatrix} cap \end{pmatrix}
$$

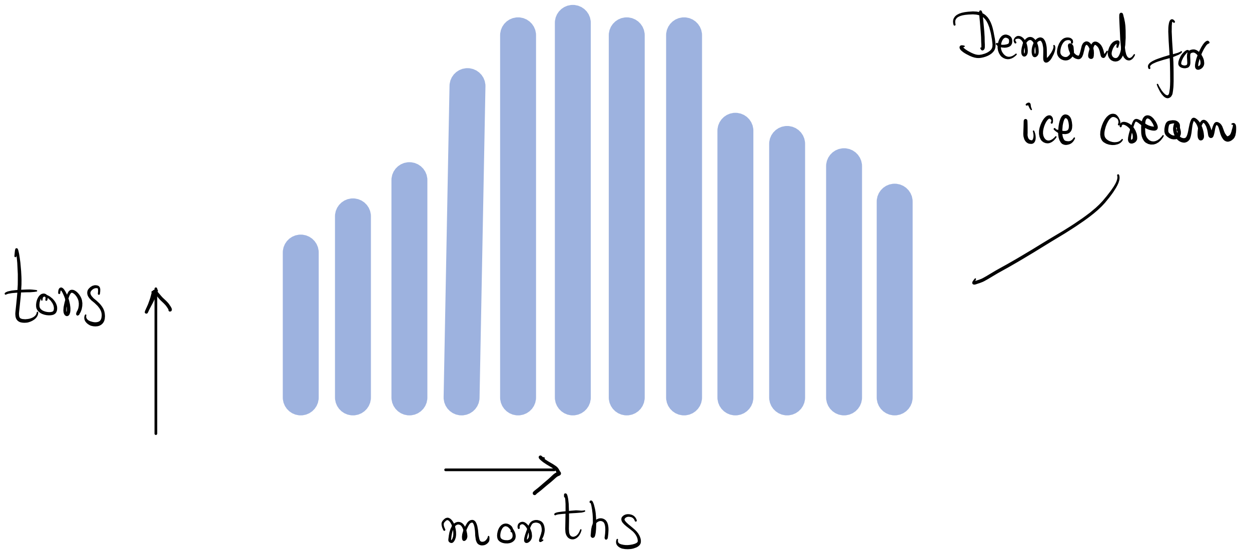
optimum $= 4$





The directions of the flows are determined by the signs of the variables.

3. Scheduling production



Given: Demands for each month $(d_1, d_2, \dots, d_{12})$

$\begin{cases} \text{Cost of changing the rate of production (A)} \\ \text{Cost of storage (B)} \end{cases}$

Variables: x_i : production in month i

S_i : surplus at the end of month i.

Constraints:

$$S_0 = 0$$
$$S_{12} = 0$$

Redundant $\rightarrow$ $x_i + S_{i-1} \geq d_i \quad (i = 1, 2, \dots, 12)$

$$x_i + S_{i-1} - S_i = d_i \quad (i = 1, 2, \dots, 12)$$
$$x_i \geq 0 \quad (i = 0, 1, 2, \dots, 12)$$
$$S_i \geq 0 \quad (i = 1, 2, \dots, 12)$$

Objective function (to minimize) $A \cdot \sum_{i=1}^{12} |x_i - x_{i-1}| + B \sum_{i=1}^{12} s_i$

Unfortunately, the objective function is not linear in the variable (x_i, s_i)

Trick 1

Introduce a new variables z_i

New constraints

$$x_i - x_{i-1} - z_i \leq 0$$

$$x_{i-1} - x_i - z_i \leq 0$$

New objective function

$$A \cdot \sum_{i=1}^{12} z_i + B \sum_{i=1}^{12} s_i$$

Trick 2

Introduce new variables u_i, w_i

u_i = increase in production

w_i = decrease in production

New constraints

$$x_i - x_{i-1} = u_i - w_i$$

$$u_i, w_i \geq 0$$

New objective function

$$A \sum_{i=1}^{12} u_i + A \sum_{i=1}^{12} w_i + B \sum_{i=1}^{12} s_i$$

Formally, one argues that

① value of original program ≤ value of the new program

② value of the new program ≤ value of the original program.

Next time

- Line fitting
- Source

Understanding and Using Linear Programming
Jiří Matoušek
Bernd Gärtner
Springer.