From linear to semidefinite

Source: Matoušek and Gärtner, Approximation Algorithms and semidefinite Programming, Chapter 2.

maximize $c^T x$

subject to $Ax = b$

$x \geq 0$

$$C = \begin{pmatrix} c_{11} & c_{1n} \\ c_{n1} & c_{nn} \end{pmatrix}$$

$$A_k = \begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix}$$

Maximize $C \bullet X$

Subject to $A_1 \circ X = b_1$

$A_2 \circ X = b_2$

$\vdots$

$A_m \circ X = b_m$

$X \succeq 0$

(Symmetric) positive semidefinite

$$\boxed{\begin{gathered} M \succeq N \\ x^T M x \geq x^T N x \end{gathered}}$$

$$B \circ X = \sum_{i,j} b_{ij} x_{ij}$$

↑

Hadamard product

$$= \operatorname{Tr} B^T X$$

$$\begin{pmatrix} b_{11} & & b_{1n} \\ | & & | \\ b_{n1} & & b_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & & x_{1n} \\ | & & | \\ x_{n1} & & x_{nn} \end{pmatrix} = \operatorname{Tr} \begin{pmatrix} b_{11} & \text{---} & b_{n1} \\ b_{1n} & \text{---} & b_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & & x_{1n} \\ | & & | \\ x_{n1} & & x_{nn} \end{pmatrix}$$

The value of a semidefinite program

$$\sup \{ C \cdot X : A(X) = 0, X \succeq 0 \}$$

finite

unbounded

Warning Even if the value is finite, it may not be attained.

Example: Maximize $-x_{11}$

subject to $x_{12} = 1$

$X \succeq 0$

Value $= 0$, not attained.
$$\begin{pmatrix} x_{11} & 1 \\ 1 & x_{22} \end{pmatrix}, \quad x_{11}x_{22} \geq 1$$

Standardization

- Inequalities
- Additional real variables

Slack variables

$$\begin{pmatrix} x_{11} & x_{12} & & x_{1n} & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ x_{n1} & & & x_{nn} & 0 \\ 0 & 0 & 0 & 0 & y_1 \end{pmatrix}$$

$A \circ X + y_1 \geq \beta$

$\vdots$

$z_i = u_i - v_i$

$u_i, v_i \geq 0$

Theorem: P: a semidefinite program

$\|X\|_F \leq R$ for all feasible X

$\|X\|_F = \sqrt{\sum_{ij} x_{ij}^2}$

Please see Thm. 2.6.1 of the text.

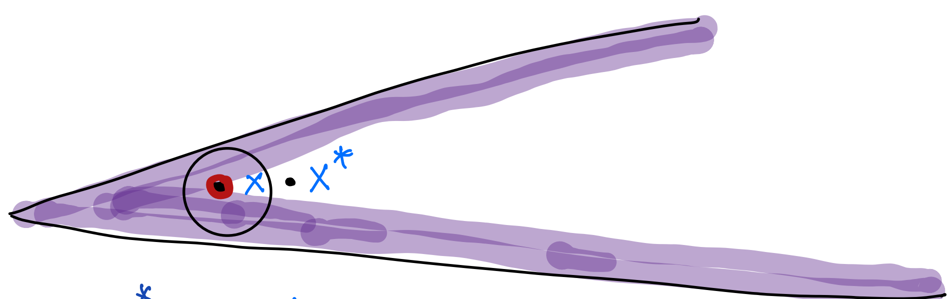
There is an algorithm that outputs

- A matrix X^* that satisfies all equality constraints, s.t. $\|X^* - X\|_F \leq \varepsilon$ for some solution X feasible and

  $C \circ X^* \geq \underset{\varepsilon}{\mathrm{Value}}(P) - \varepsilon$

Running time depends on $\log(R/\varepsilon)$

- An ellipsoid E whose points satisfy all equality constraints, E contains all feasible points, and has volume less than an ε-ball's.



- $\mathrm{val}(X^*) \geq \mathrm{val}(Y) - \varepsilon \quad \forall Y$ that is ε-deep.
- $\|X - X^*\|_F \leq \varepsilon$.

Positive semidefinite matrices $M \in \mathbb{R}^{n \times n}$

- Symmetric $M^T = M$

  Such matrices have

  - an orthonormal basis of eigenvectors
  - with corresponding real eigenvalues

- All eigenvalues are non-negative

Theorem: Assume $M \in \mathbb{R}^{n \times n}$ is symmetric.

M is psd $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$

$\Leftrightarrow \quad x^T M x \geq 0 \quad \forall x \in \mathbb{R}^n$

$\Leftrightarrow \quad M = B^T B$ for some $B \in \mathbb{R}^{n \times n}$

PSD_n : set of all positive semidefinite matrices $M \in \mathbb{R}^{n \times n}$.

$M \in PSD_n$ is positive definite if $x^T M x > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

M is positive definite

$\Leftrightarrow \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$

$\Leftrightarrow \quad M = B^T B$ for some non-singular $B \in \mathbb{R}^{n \times n}$

Cholesky factorization

Given: $M \in \mathbb{R}^{n\times n}$ symmetric

Task: Output a non-singular B s.t. $M = B^T B$

OR

Output $x \in \mathbb{R}^n$, $x \neq 0$ s.t. $x^t M x \leq 0$

Idea: Gaussian elimination

$$M = U^T D U$$

$$= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \underset{d_i > 0}{\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{B}$$

o $M_1 = M = \begin{bmatrix} m_{11} & & \\ & & \\ & & \end{bmatrix}$ if $m_{11} \leq 0$, then $x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

M is not positive definite

m_{11} a_1 a_j a_n

a_1

$- \; a_i$ $\quad m_{ij} - \frac{a_i a_j}{m_{11}}$

a_{n-1}

$$E_1 M = \begin{bmatrix} m_{11} & & \\ 0 & & \\ 0 & & \\ 0 & & \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

symmetric

?

$$E_1 M E_1^T = \underbrace{\begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{bmatrix}}_{M_2}$$

positive definite?

$$\underbrace{E_n \cdots E_2 E_1}_{B^T} M \underbrace{E_1^T E_2^T \cdots E_n^T}_{B} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

upper triangular

$$\begin{array}{cc} & \begin{array}{ccccc} & & & i & \end{array} \\ E_i = & \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & 1 & \\ & & & | & \ddots \\ & & & & 1 \end{bmatrix} \end{array}$$

non-singular

- If we encounter a $d_i \leq 0$, then

$$y^T \; E_{i-1} \cdots E_1 M E_1^T \cdots E_{i-1}^T y = y^T \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & \boxed{d_i} \end{bmatrix} y$$

Set $y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - i$ and $x = \underbrace{E_1^T \cdots E_{i-1}^T y}_{\text{non-zero}}$

Then, $x^T M x = y^T M_i y \leq 0.$

- Otherwise, at the end, we have

$$B^T M B = D$$

$$M = \underbrace{(B^T)^{-1}}_{\text{lower triangular}} D \underbrace{B^{-1}}_{\text{upper triangular } (U)}$$

$$= U^T D U$$

For PSD, repeat the same argument.

Observe, if some $d_i = 0$, then the i^{th} row and i^{th} column of M_i must already be 0. (Why?)

[Hint: Suppose

$$M_i = \begin{array}{cc} & \begin{array}{cccc} & & & j \end{array} \\ \begin{array}{c} \\ \\ \\ j \end{array} & \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & d_i & \alpha \\ & & \alpha & \beta \end{pmatrix} \end{array}$$

Examine the 2×2 matrix $\begin{pmatrix} 0 & \alpha \\ \alpha & \beta \end{pmatrix}$.

Assume $\alpha \neq 0$, and obtain a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ s.t.

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} < 0. \quad]$$