Last time

- The Lovász theta function
- The Shannon capacity of a graph
- The zero-error capacity of the 5/2 channel

The Lovász theta function

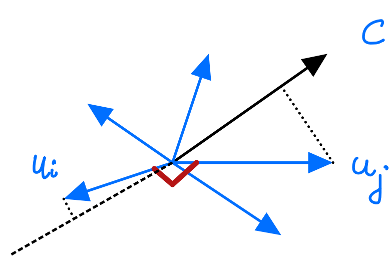
$$\vartheta(G) = \min_{\substack{c \\ \|c\|=1}} \max_{i \in [n]} \frac{1}{\langle c, u_i \rangle^2}$$

The minimum exists! See the textbook.

each vertex is assigned a unit vector → $u_1, u_2, \ldots, u_n$

$\|u_i\| = 1$

$u_i \perp u_j$ if $\{i,j\} \notin E$ ← non-adjacent vertices take up disjoint portions of the budget



i and j are non-adjacent

$(u_1, u_2, \ldots, u_n, c)$: an orthogonal representation of G.

c : handle

Claim: $\Theta(G \otimes H) \le \Theta(G)\,\Theta(H)$

This holds with equality. We will see later.

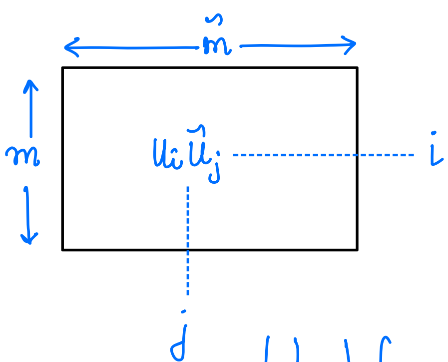
Suppose $(u_1, \dots, u_m, c) \in \mathbb{R}^M$ achieve $\Theta(G)$

and $(\tilde{u}_1, \dots, u_{\tilde{m}}, \tilde{c}) \in \mathbb{R}^{\tilde{M}}$ achieve $\Theta(H)$.

We will build an orthogonal representation for $G \otimes H$ out of these vectors.

Idea: Tensolization

$$U = u \otimes \tilde{u} \in \mathbb{R}^{m \times \tilde{m}}, \qquad u \in \mathbb{R}^m, \quad \tilde{u} \in \mathbb{R}^{\tilde{m}}$$



Suppose $\left.\begin{array}{l} U = u \otimes \tilde{u} \\ W = w \otimes \tilde{w} \end{array}\right\} \in \mathbb{R}^{m \times m'}$

$$U \cdot W = \sum_{ij} u_i \tilde{u}_j \, w_i \tilde{w}_j$$

$$= \Big(\sum_i u_i w_i\Big)\Big(\sum_j \tilde{u}_j \tilde{w}_j\Big)$$

$$= \langle u_i, w_i \rangle \langle \tilde{u}_j, \tilde{w}_j \rangle$$

Then, $(u_i \otimes \tilde{u}_j : i = 1, 2, \dots, m,\ j = 1, 2, \dots, \tilde{m},\ c \otimes \tilde{c})$ is an orthogonal representation for $G \otimes H$. **(Check!)**

$$(c \otimes \tilde{c}) \cdot (u_i \otimes \tilde{u}_j) = \langle c, u_i \rangle \langle \tilde{c}, \tilde{u}_j \rangle \Rightarrow \Theta(G \otimes H) \le \Theta(G)\,\Theta(H).$$

Today:

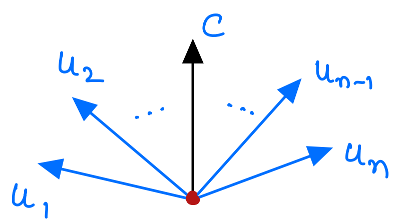
- Vector k-colouring of G.
- Colouring 3-colourable graphs.

Suppose $(u_1, u_2, \dots, u_n, c)$ is an orthonormal representation of G.

$$\begin{pmatrix} - & u_1 & - \\ - & u_2 & - \\ & \vdots & \\ - & u_n & - \\ - & c & - \end{pmatrix} \begin{pmatrix} | & | & & | & | \\ u_1 & u_2 & \cdots & u_n & c \\ | & | & & | & | \end{pmatrix}$$

$$= \begin{array}{cc} & \\ i - & \begin{pmatrix} & & \\ & \langle u_i, u_j \rangle & \langle u_i, c \rangle \\ & & \\ & \langle c, u_j \rangle & \end{pmatrix} \\ & \begin{array}{c} | \\ j \end{array} \end{array}$$

- if $(i,j) \notin E$, then $\langle u_i, u_j \rangle = 0$
- if the objective value is t, then $|\langle c, u_i \rangle| \geq \frac{1}{\sqrt{t}}$

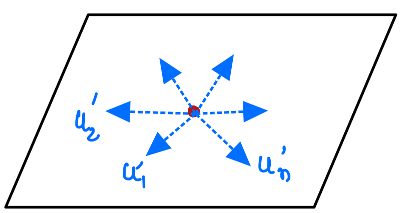


CLAIM: If this configuration is optimal, then $|\langle u_i, c \rangle| = 1/\sqrt{t} \quad \forall i$.

Then,

$$\|u_i'\|^2 = 1 - \frac{1}{t} = \frac{t-1}{t}$$

$$\{i,j\} \notin E \Rightarrow \langle u_i', u_j' \rangle = -\frac{1}{t}$$



projections of u_i onto $c^\perp$

Another program for $\vartheta(G)$

$$\boxed{\begin{aligned} &\text{minimize} \quad t \\ &\text{subject to} \quad y_{ij} = -1/(t-1) \quad \text{if } (i,j) \notin E \\ &\qquad\qquad\quad\; y_{ii} = 1 \\ &\qquad\qquad\quad\; Y \succeq 0 \end{aligned}}$$

Idea: y_{ij} represents $\langle u_i', u_j' \rangle$ after the vectors have been normalized.

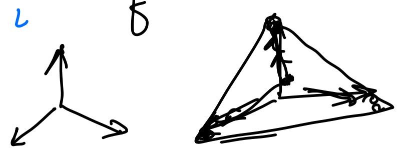
Vector k-colouring (of the complement graph)

An assignment $\gamma : V \to S^{n-1}$ (unit vectors in $\mathbb{R}^n$) such that $\langle \gamma(v), \gamma(w) \rangle = -\frac{1}{k-1}$, $\{v,w\} \in E$. since we work with the complement

[$k \in \mathbb{R}$, $k > 1$; adjacent vertices receive vectors that point away from each other.]

G is k-colourable $\Rightarrow$ G has a vector k-colouring.

$$\gamma(v) = \sqrt{\frac{k}{k-1}} \begin{pmatrix} -1/k \\ -1/k \\ \frac{k-1}{k} \\ \vdots \\ -1/k \end{pmatrix} \leftarrow i \quad \text{if } v \text{ has colour } i.$$



Then, $\gamma(v)\gamma(w) = -1/(k-1)$ whenever $\{v,w\} \in E$. **(Check!)**

If G is 3-colourable, then G has a $\underbrace{\text{vector } 3\text{-colouring}}$.

Can be found by solving a semidefinite program

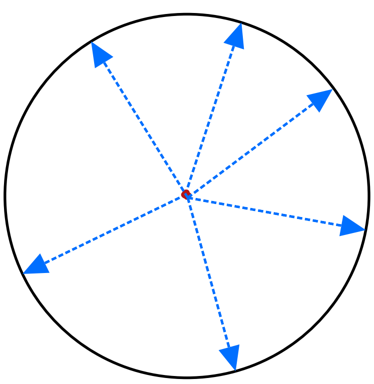
Karger-Motwani-Sudan

G + vector 3-colouring $\rightsquigarrow \approx n^{1/3}$ colouring

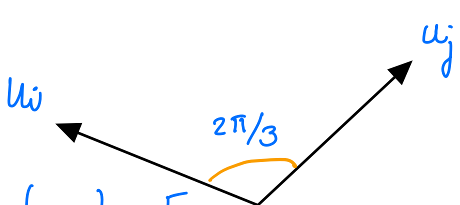
- Given a vector 3-colouring, obtain an independent set of size $\approx n^{2/3}$.
- Repeatedly find independent sets and assign a new colour to each.

$$\boxed{\#\text{ colours } \leq \int_1^n \frac{1}{x^{2/3}}\,dx = 3x^{1/3}\Big|_1^n \leq 3n^{1/3}}$$

KMS rounding



- Solve the semidefinite program to obtain a vector 3-colouring.
- Let I be the set of vertices whose vectors fall in the halfspace
$$\{x : \hat{r}\cdot x \geq t\}$$
where $\hat{r}$ is random (!)
$$t \approx \left(\frac{2}{3}\ln n\right)^{1/2}$$



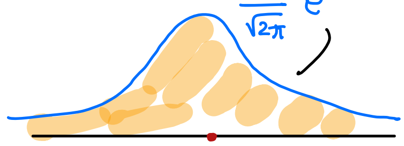
$\{i, j\} \in E$

- If an edge falls in I remove both its vertices

What is random?

$r = (r_1, r_2, \dots, r_n) \in \mathbb{R}^n$, where each $r_i \sim N(0,1)$, independently.

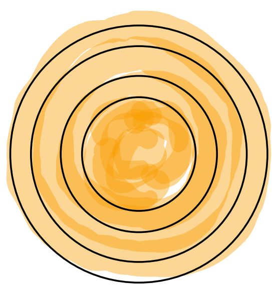
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



density function of $N(0,1)$

- r is spherically symmetric.
- For every unit vector $\hat{v}$,

$$\hat{v} \cdot r \sim N(0,1).$$



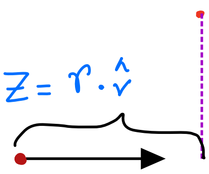
density

Spherically symmetric.

n-dimensional standard normal Gaussian

r

$z = r \cdot \hat{v}$



$\hat{v}$ unit vector

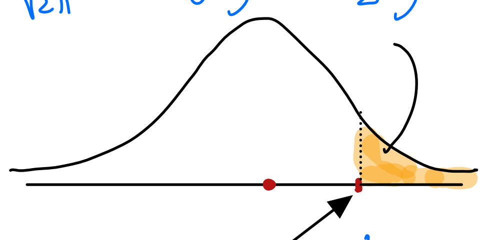
Is I large?

$$\Pr[v \in I] = \Pr[z \geq t]$$

$$E[|I|] \geq n \cdot n^{-1/3} \approx n^{2/3}$$

Linearity of expectation

$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{t} - \frac{1}{t^3} \right) \exp\left(\frac{-t^2}{2} \right)$

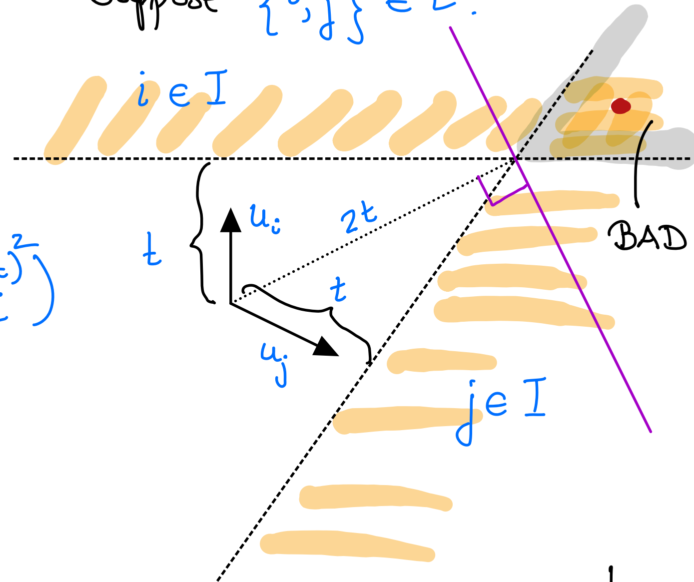


$t = \left(\frac{2}{3} \ln n \right)^{1/2}$

Tail probability $\approx \frac{1}{\sqrt{2\pi}} \frac{1}{\left(\frac{2}{3} \ln n \right)^{1/2}} \cdot n^{-1/3}$

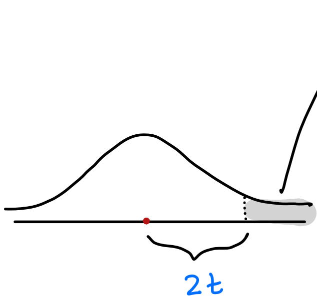
Does the vertex survive?

Suppose $\{i,j\} \in E$.



$$\Pr[\{i,j\} \subseteq I]$$

$$\lesssim \frac{1}{\sqrt{2\pi}}\,\frac{1}{2t}\exp\left(-\frac{(2t)^2}{2}\right)$$



union b

$$\Pr[v \text{ survives in } I] = \frac{1}{\sqrt{2\pi}}\left(\left(\frac{1}{t}-\frac{1}{t^3}\right)e^{-t^2/2} - \frac{n}{2t}e^{-4t^2/2}\right)$$

$$t = \left(\frac{2}{3}\ln n\right)^{1/2} \geq 2$$

$$\lesssim \frac{1}{\sqrt{2\pi}}\,\frac{1}{2t}\left(\frac{3}{2}e^{-\frac{\ln n}{3}} - n e^{-\frac{4\ln n}{3}}\right)$$

$$\frac{3}{2}n^{-1/3} - n\cdot n^{-\frac{4}{3}}$$

$$\boxed{\frac{3}{2}n^{-1/3} - n^{-1/3}}$$

$$\lesssim \frac{1}{\sqrt{\ln n}}\, n^{-1/3}$$

$$E[|\text{independent set}|] \lesssim \frac{n^{2/3}}{\sqrt{\ln n}}$$

With prob. at least $\frac{n^{-1/3}}{2\sqrt{\ln n}}$, the independent set has size at least $\frac{n^{2/3}}{2\sqrt{\ln n}}$.

There are n-vertex graphs G that have a vector 3-colouring, but for which every proper colouring requires n^{δ} colours (for a $\delta > 0$, independent of n). See Section 9.5.