Conic Programming and Duality (Lecture 18)

Separation theorem. Let $K \subseteq V$ be a closed convex cone and let $b \in V \setminus K$. Then there exists $y \in V$ s.t.

$$(y, x) \geq 0 \text{ for all } x \in K \text{ and } (y, b) < 0.$$

The separation theorem when applied to

$K = \{Ax : x \in \mathbb{R}^n_+\} \subseteq V = \mathbb{R}^m$ (here A is an $m \times n$ matrix)

implies the Farkas lemma in the LP setting.

For any $b \in \mathbb{R}^m$, exactly one of the following holds:

- either $Ax = b$, $x \geq 0$ has a solution $x \in \mathbb{R}^n$
- or $\underbrace{A^T y \geq 0}$, $b^T y < 0$ has a solution $y \in \mathbb{R}^m$.

The separation theorem implies that if the first system has no solution then there exists $y \in \mathbb{R}^m$ s.t. $\underbrace{y^T A x \geq 0 \;\; \forall x \in \mathbb{R}^n_+}$ and $y^T b < 0$.

This means $A^T y \in (\mathbb{R}^n_+)^* = \mathbb{R}^n_+$, i.e, $A^T y \geq 0$.

Our goal now is to generalize the Farkas lemma to deal with systems of the form $Ax = b$, $x \in K$.

We would like to be able to claim the following:

Let $K \subseteq V$ be a closed convex cone, and let $b \in W$. Then

- either $A(x) = b$, $x \in K$ has a solution $x \in V$
- or the system $A^T(y) \in K^*$, $(b, y) < 0$ has a solution,

but not both.

This is a straightforward generalization of the Farkas lemma in the LP setting. However the above claim is not true.

Consider the following example: K = toppled ice cream cone in $\mathbb{R}^3$, and let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$.

The first system asks for a point $(x, y, z) \in K$ such that $y = 0$ and $z = 1$. There is no such point since $xy \geq z^2$ is one of the constraints for K.

The second system asks for a point $(\alpha, \beta) \in \mathbb{R}^2$ such that $(\alpha \;\; \beta)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{(0 \;\; \alpha \;\; \beta) \in K^*}_{\text{so } \beta^2 \leq 0 \Rightarrow \beta = 0}$

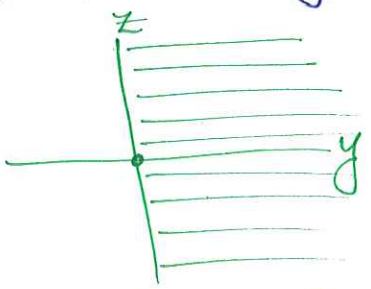
and $((0,1), (\alpha, \beta)) = \underline{\beta < 0}$.

Thus both the systems are unsolvable.

So what is going wrong here? To apply the separation theorem, the cone $C = A(K) = \{A(x) : x \in K\}$ needs to be closed. Observe that this is not true in our example. Here C is the projection of K onto the y-z plane. That is,

$$C = \{(y, z) \in \mathbb{R}^2 : (x, y, z) \in K\} = \{\vec{0}\} \cup (\{y \in \mathbb{R} : y > 0\} \times \mathbb{R})$$

Thus C is



where the only point on z-axis that belongs to C is the origin. So C is <u>not closed</u>.

Hence we cannot apply the separation theorem on C.

To save the situation, we work with the <u>closure</u> of C.

The closure $\overline{C}$ of C is the set of all limit points of C. Formally, $b \in \overline{C}$ if and only if there exists a sequence $(y_k)_{k \in \mathbb{N}}$ s.t. $y_k \in C$ for all k and $\lim_{k \to \infty} y_k = b$.

When C is a convex cone, $\overline{C}$ is a closed convex cone.

Definition. Let $K \subseteq V$ be a closed convex cone. The system $A(x) = b$, $x \in K$ is called limit-feasible if there exists a sequence $(x_k)_{k \in N}$ such that $x_k \in K$ for all $k \in N$ and $\lim_{k \to \infty} A(x_k) = b$.

So if $A(x) = b$, $x \in K$ is limit-feasible then $b \in \overline{C}$ where $C = \{A(x) : x \in K\}$. Moreover, if $b \in \overline{C}$ then the system $A(x) = b, x \in K$ is limit-feasible. In more detail, if $(y_k)_{k \in N}$ is a sequence in C converging to b then any sequence $(x_k)_{k \in N}$ s.t. $y_k = A(x_k)$ for all k proves the limit-feasibility of $A(x) = b$, $x \in K$.

The correct Farkas lemma (cone version)

Let $K \subseteq V$ be a closed convex cone and let $b \in W$.
- Either the system $A(x) = b$, $x \in K$ is limit-feasible
- Or the system $A^T(y) \in K^*$, $(b, y) < 0$ has a solution

but not both.

Proof. If $A(x) = b$, $x \in K$ is limit-feasible then we choose any sequence $(x_k)_{k \in N}$ that proves its limit feasibility. For $y \in W$, we compute

$$(y, b) = (y, \lim_{k \to \infty} A(x_k)) = \lim_{k \to \infty} (y, A(x_k)) = \lim_{k \to \infty} (A^T(y), x_k).$$

If $A^T(y) \in K^*$ then $x_k \in K$ implies that $(A^T(y), x_k) \geqslant 0$ for all $k \in \mathbb{N}$. Thus we get $(y, b) \geqslant 0$. In other words, the second system has no solution.

Now suppose $A(x) = b$, $x \in K$ is not limit-feas. This is the same as saying $b \notin \overline{C}$ where $C = \{A(x) : x \in K\}$. Since $\overline{C}$ is a closed convex cone, we can apply the separation theorem and obtain a hyperplane that strictly separates b from $\overline{C}$ (and thus from C).

This means there exists $y \in W$ s.t. $(y, b) < 0$ and for all $x \in K$, $(y, A(x)) = \underbrace{(A^T(y), x) \geqslant 0}_{\text{Equivalently, } A^T(y) \in K^*}$.

Cone Programs

A cone program is an optimization problem of the form
$$\begin{aligned} \max\ & (c, x) \\ \text{s.t.}\ & A(x) = b \\ & x \in K \quad (K \text{ is a closed convex cone}) \end{aligned}$$

Note that we can be more general and say it is an optimization problem of the form
$$\begin{aligned} \max\ & (c, x) \\ \text{s.t.}\ & b - A(x) \in \mathcal{L} \quad (\text{both } \mathcal{L} \text{ and } K \text{ are closed convex cones}) \\ & x \in K \end{aligned}$$

For the sake of simplicity, let us take $\mathcal{L} = \{\vec{0}\}$.

So our primal program is the following:

$$\begin{aligned} &\max \ (c, x) \\ &\text{s.t.} \quad A(x) = b \\ &\qquad\quad x \in K. \end{aligned}$$

Given a feasible sequence $(x_k)_{k\in N}$ of the above cone program, we define its value as

$$(c, (x_k)_{k\in N}) = \limsup_{k\to\infty} (c, x_k)$$

The limit value of the above cone program is defined as $\sup\{(c, (x_k)_{k\in N}) : (x_k)_{k\in N}$ is a feasible sequence of the cone program$\}$.

What is the dual program of the above primal program? Taking a cue from duality in linear programming, let us write it as:

$$\begin{aligned} &\min \ (b, y) \\ &\text{s.t.} \quad A^T(y) = c + s \\ &\qquad\quad s \in K^* \end{aligned}$$

We can write this more succinctly as $A^T(y) - c \in K^*$.

Let us start with the weak duality theorem.

Theorem. If the dual program is feasible and if the primal program is limit-feasible then the limit value of the primal program $\leqslant$ the value of the dual program.

Proof. Let y be any feasible solution of the dual program and let $(x_k)_{k\in N}$ be any feasible sequence of the primal program. We have:

$$0 \leq (\underbrace{A^T(y) - c}_{\in K^*}, \underbrace{x_k}_{\in K}) = (A^T(y), x_k) - (c, x_k)$$

$$= (y, A(x_k)) - (c, x_k) \text{ for } k \in \mathbb{N}.$$

Hence $\limsup\limits_{k\to\infty} (c, x_k) \leq \limsup\limits_{k\to\infty} (y, A(x_k))$

$$= \lim_{k\to\infty} (y, A(x_k)) = (y, b).$$

Since the feasible sequence $(x_k)_{k\in N}$ was arbitrary, this means that the limit value of the primal program is at most (y, b). Since y was an arbitrary feasible solution of the dual program, the thm follows. ▨

Regular Duality

Theorem. The dual program is feasible and has a finite value β if and only if the primal program is limit-feasible and has a finite limit value Γ. Moreover, $\beta = \Gamma$.

Before we prove the above theorem, let us introduce some notation. Let V and W be real and finite-dimensional vector spaces.

$V \oplus W$, the direct sum of V and W, is the set $V \times W$, turned into a vector space with scalar product via $(x, y) + (x', y') = (x + x', y + y')$, $\lambda(x, y) = (\lambda x, \lambda y)$, and $((x,y),(x',y')) = \underbrace{(x,x')}_{\text{scalar product}} + \underbrace{(y,y')}_{\text{scalar product}}$.

<u>Fact.</u> Let $K \subseteq V$ and $\mathcal{L} \subseteq W$ be closed convex cones. Then $K \oplus \mathcal{L} = \{(x, y) \in V \oplus W : x \in K, y \in \mathcal{L}\}$ is again a closed convex cone.

<u>Claim.</u> Let $K \subseteq V$ and $\mathcal{L} \subseteq W$ be closed convex cones. Then $(K \oplus \mathcal{L})^* = K^* \oplus \mathcal{L}^*$.

<u>Proof of Regular Duality.</u> If the dual program is feasible and has value β, then we have:

$$A^T(y) - c \in K^* \Rightarrow \underbrace{(b, y)}_{\text{the inner product}} \geq \beta. \qquad \text{— (1)}$$

We also have $A^T(y) \in K^* \Rightarrow (b, y) \geq 0$. — (2)

Otherwise we could add a large positive multiple of y to any dual feasible solution and obtain a dual feasible solution of value smaller than β.

We can now merge (1) and (2) into the single implication: $A^T(y) - zc \in K^*, z \geq 0 \Rightarrow \langle b, y \rangle \geq z\beta$.

For $z > 0$, we obtain the above implication from (1) by multiplying all terms with z and calling "zy" y. For $z = 0$, this is simply (2).

In order to apply Farkas lemma, let us rewrite the above implication as:

$$\left(\begin{array}{c|c} A^T & -c \\ \hline 0 & 1 \end{array}\right)(y, z) \in K^* \oplus \mathbb{R}_+ \Rightarrow \langle (b, -\beta), (y, z) \rangle \geq 0. \qquad \text{— (3)}$$

We are now ready to apply the Farkas lemma. We are precisely in the situation where the second system has no solution. So implication (3) holds if and only if

$$\left(\begin{array}{c|c} A & 0 \\ \hline -c^T & 1 \end{array}\right)(x, z) = (b, -\beta), \quad (x,z) \in (K^* \oplus \mathbb{R}_+)^* = K \oplus \mathbb{R}_+$$

is limit-feasible. This system is limit-feasible if and only if there are sequences $(x_k)_{k\in\mathbb{N}}, (z_k)_{k\in\mathbb{N}}$ with $x_k \in K$, $z_k \geq 0$ for all k such that

$$\underbrace{\lim_{k\to\infty} A(x_k) = b}_{\text{This shows that the primal program is limit-feasible}} \quad \text{and} \quad \underbrace{\lim_{k\to\infty} \langle c, x_k\rangle - z_k = \beta.}_{\text{This shows that the limit value of the primal program } \geq \beta.}$$

Moreover, weak duality shows that the limit value of the primal program $\leq \beta$. This proves the "only if" direction.

The "if" direction. Let the primal program be limit-feasible with finite limit value γ and suppose the dual program is infeasible. This implies:

$$A^T(y) - zc \in K^* \Rightarrow z \leq 0,$$

since if (y, z) violates it, then $\frac{1}{z}\cdot y$ would be a dual feasible solution. Let us write the above implication as:

$$\left(A^T \,\middle|\, -c\right)(y, z) \in K^* \Rightarrow \langle (0,-1), (y, z)\rangle \geq 0.$$

This means that the system $\left(\begin{array}{c} A \\ \hline -c^T \end{array}\right)(x) = (0,-1),\ x \in (K^*)^* = K$ is limit-feasible. This means that there are sequences $(x_k)_{k\in\mathbb{N}}$ with $x_k \in K$ for all k such that

$$\lim_{k\to\infty} A(x_k) = 0 \quad \text{and} \quad \lim_{k\to\infty} \langle c, x_k\rangle = 1.$$

But this is a contradiction as elementwise addition of $(x_k)_{k\in\mathbb{N}}$ to any feasible sequence of the primal program that attains the limit value γ would result in a feasible sequence that witness limit value $\geq \gamma + 1$. Thus the dual program must have been feasible. Easy to show $\beta = \gamma$. ▨