Conic Programming and Duality (Lecture 19) ①

A cone program is an optimization problem of the form

$$\max \langle c, x \rangle \quad \text{s.t.} \quad A(x) = b, \quad x \in K.$$

Recall that A is a linear operator from V to W. $K \subseteq V$ is a closed convex cone and $c \in V$, $b \in W$.

The value of a feasible cone program is defined as

$$\sup\{\langle c, x\rangle : Ax = b,\ x \in K\}.$$

The limit value of a limit-feasible cone program is defined as $\sup\{\underbrace{\langle c, (x_k)_{k\in\mathbb{N}}\rangle}_{= \limsup_{k\to\infty} \langle c, x_k\rangle} : \lim_{k\to\infty} A(x_k) = b,\ x_k \in K \text{ for all } k\}$.

It is immediate from the definitions that the limit value of a limit-feasible cone program $\geqslant$ its value. Moreover, this inequality might be strict.

For example,

$$\max z \quad \text{s.t.} \quad z \leqslant 1,\ x = 0,\ (x, y, z) \in K_{top}$$

(the toppled icecream cone in $\mathbb{R}^3$)

Rewriting this in the above form,

$$\max z \quad \text{s.t.} \quad z + s = 1,\ x = 0,\ (x, y, z, s) \in K_{top} \oplus \mathbb{R}_+$$

The value of the above cone program is 0. However, its limit value is 1 due to the limit-feasible seq. $\left(\frac{1}{k}, k, 1, 0\right)_{k\in\mathbb{N}}$.

Primal program	Dual program
$\max \ \langle c, x \rangle$	$\min \ \langle b, y \rangle$
s.t.	s.t.
$A(x) = b$	$A^T(y) = c + s$
$x \in K$.	$s \in K^*$

$\left.\begin{array}{l} A^T(y) = c + s \\ s \in K^* \end{array}\right\}$ Equivalently, $A^T(y) - c \in K^*$.

<u>Weak duality.</u> If the dual program is feasible and if the primal program is limit-feasible then the limit value of the primal program $\leqslant$ the value of the dual program.

<u>Regular duality.</u> The dual program is feasible and has a finite value β if and only if the primal program is limit-feasible and has a finite value Γ. Moreover, $\beta = \Gamma$.

So regular duality implies the following picture:

Value of the primal = Limit value of the dual   Limit value of the primal = Value of the dual

As seen in the earlier example, there can be a gap between the value of a feasible cone program and its limit value.

<u>Strong duality.</u> If the primal program is feasible, has a finite value Γ and has an interior point $\tilde{x}$, then the dual program is also feasible and has the same value Γ.

To prove strong duality, let us recall the following theorem (see the notes of Lecture 16):

<u>Definition.</u> An interior point of our primal program is a point $\underbrace{x \in \operatorname{int}(K)}$ such that $A(x) = b$.

that is, there is a sufficiently small ball around x fully contained in K.

<u>Theorem.</u> If the primal program has an interior point (which also means it is feasible), then the value equals the limit value.

<u>Proof of strong duality.</u> The primal program is feasible $\Rightarrow$ it is limit-feasible. It has an interior point $\Rightarrow$ its value $=$ limit value. Let us now apply regular duality.

The primal program is limit-feasible and has a finite limit value γ $\Rightarrow$ the dual program is feasible and has a finite value β; moreover, $\beta = \gamma$. ▨

<u>Semidefinite Programming Case</u>

<u>Primal program:</u> $\max \; C \bullet X$

$$\text{s.t.} \quad A_i \bullet X = b_i, \quad i = 1, \dots, m$$
$$X \succeq 0$$

(Recall the notation $C \bullet X = \sum_{i,j=1}^{n} c_{ij} x_{ij}$.)

<u>Dual program:</u> $\min \; b^T y$

$$\text{s.t.} \quad \sum_{i=1}^{m} y_i A_i - C \succeq 0.$$

Theorem. If the primal program is feasible and has a finite value γ, and if there is a positive definite* matrix $\tilde{X}$ s.t. $A(\tilde{X}) = b$, then the dual program is feasible and has finite value $\beta = \gamma$.

(*: a positive definite matrix will be an interior point in PSD_n)

The above theorem immediately follows from strong duality of cone programs by taking $V = SYM_n$, $W = \mathbb{R}^m$, $K = PSD_n$. We have seen that $K^* = PSD_n$ (this cone is self-dual). We also know that

$$A^T(y) = \sum_{i=1}^{m} y_i A_i.$$

Thus the above theorem follows.

A useful lemma: Let M be an $n \times n$ real matrix. We have $M \succcurlyeq 0$ if and only if there are unit-length vectors $u_1, \ldots, u_n \in S^{n-1}$ and non-negative real numbers $\lambda_1, \ldots, \lambda_n$ such that $M = \sum_{i=1}^{n} \lambda_i u_i u_i^T$.

Proof. If there are such vectors $u_1, \ldots, u_n \in S^{n-1}$ and non-negative real numbers $\lambda_1, \ldots, \lambda_n$ s.t. $M = \sum_{i=1}^{n} \lambda_i u_i u_i^T$ then $v^T M v$ where $v \in \mathbb{R}^n$ equals

$$\sum_{i=1}^{n} \lambda_i \underbrace{(v^T u_i)}_{\text{a real no.}} \underbrace{(v^T u_i)^T}_{\text{the same no.}} = \text{a conic comb. of squares} \geq 0.$$

Thus $M \succcurlyeq 0$.

For the "only if" direction, let us diagonalize $M = SDS^T$ where S is an orthogonal matrix (whose columns are the eigenvectors of M)

and D is a diagonal matrix with non-negative values $\lambda_1, \ldots, \lambda_n$ (these are M's eigenvalues) on its diagonal.

Define u_i = i-th column of S.

So $M = \sum_{i=1}^{n} \lambda_i u_i u_i^T$ since $M = S\left(\sum_{i=1}^{n} D^{(i)}\right) S^T$

Thus $M = \sum_{i=1}^{n} S D^{(i)} S^T$ | where $D^{(i)}$ is the all-zeros matrix with only one non-zero entry which is its (i,i)-th entry $= \lambda_i$.

$= \sum_{i=1}^{n} \lambda_i u_i u_i^T$. By the orthogonality of S, $\|u_i\| = 1$ for all i. ▨

<u>An application of the strong duality theorem</u>

<u>Theorem.</u> Let $C \in SYM_n$. Then the largest eigenvalue of C is $\lambda = \max\{x^T C x : x \in \mathbb{R}^n, \|x\| = 1\}$.

<u>Proof.</u> Observe that $x^T C x = C \bullet x x^T$ and $\|x\| = 1$ is the same as $Tr(x x^T) = 1$. Thus λ is the value of the <u>constrained optimization problem</u>: $\max\ C \bullet x x^T$

s.t. $Tr(x x^T) = 1$.

<u>Claim.</u> Positive semidefinite matrices of rank 1 are exactly the ones of the form $x x^T$ for some vector x. (This is an exercise.)

Consider the following SDP: $\max\ C \bullet X$

s.t. $Tr(X) = 1$

$X \succcurlyeq 0$.

This SDP along with the constraint that X has rank 1 is equivalent to the above constrained optimization problem (using the above claim).

Thus the above SDP is a relaxation of the above constrained optimization problem

Recall that any psd matrix X can be written as

$$X = \sum_{i=1}^{n} \mu_i x_i x_i^T \text{ where } \mu_i \geq 0 \text{ for all } i$$
$$\text{and } \|x_i\| = 1 \text{ for all } i.$$

So $Tr(x_i x_i^T) = 1$ and if X is a feasible solution of our SDP then $Tr(X) = 1$. So

$$Tr(X) = 1 = \sum_{i=1}^{n} \mu_i Tr(x_i x_i^T) = \sum_{i=1}^{n} \mu_i.$$

We have $C \cdot X = C \cdot \sum_{i=1}^{n} \mu_i x_i x_i^T$

$$= \sum_{i=1}^{n} \mu_i (C \cdot x_i x_i^T) \leq \sum_{i=1}^{n} \mu_i \left(\max_i C \cdot x_i x_i^T\right)$$

$$= \max_i C \cdot x_i x_i^T \leq \lambda$$

(Since these vectors x_i are feasible solutions of the first optimization prob.)

We also have $C \cdot X^* \geq \lambda$ for some X^* since the SDP is a relaxation of this optimization problem. Hence both the programs have the same value λ.

Now we will use the strong duality theorem for SDP. Since $Tr(X) = \underbrace{I_n}_{\text{identity matrix}} \cdot X$, $A^T(y) = y I_n$.

So the dual cone program is $\min y$

s.t. $y I_n - C \succeq 0$.

Since the primal program has a feasible solution that is positive definite (let $X = \frac{1}{n} I_n$, for example), the strong duality theorem applies – so the value of the above SDP is also λ. But what is λ?

If C has eigenvalues $\lambda_1, \dots, \lambda_n$, then $y I_n - C$ has eigenvalues $y - \lambda_1, \dots, y - \lambda_n$ and the constraint $y I_n - C \succeq 0$ requires all of them to be ≥ 0. The smallest y for which $y - \lambda_i \geq 0$ for all i is $y = \lambda_1$, i.e, $y =$ the Largest eigenvalue of C. This proves the theorem. ∎