27 Jan 2022

Last time

Linear programs

- objective function
- constraints
- feasible solution
- optimal solution
- infeasible, unbounded LPs
- system of linear equations versus linear programs.

Examples

- Diet problem
- Network flows
- Scheduling production
- Line fitting

4 fitting a line

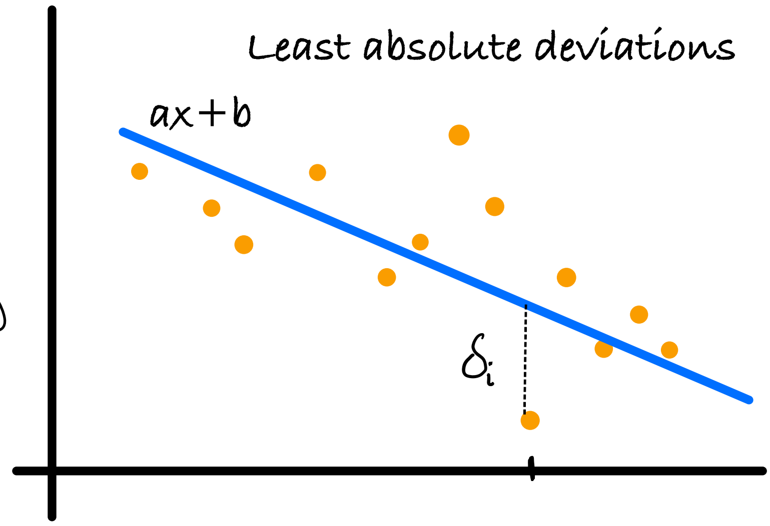
Given: n points

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Goal: find a line $ax+b$ s.t.

$$\sum_{i=1}^{n} |y_i - (ax_i + b)|$$

is minimum



A different viewpoint

The set of lines is the subspace

$$\left\{ a\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + b\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

The LP

$$\min \sum_{i=1}^{n} \delta_i$$

Subject to (for $i=1,2,\ldots,n$)

$$x_i a + b - \delta_i \leq y_i$$

$$-x_i a - b - \delta_i \leq -y_i$$

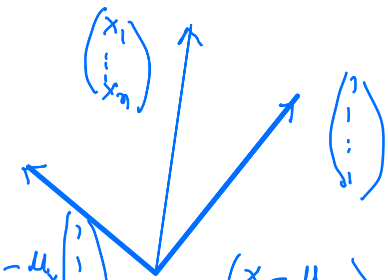
$$\delta_i \geq 0$$

$n+2$ variables

The LP allows us to find the point in this subspace that is closest to $\bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ \\ y_n \end{pmatrix}$.

where distance$(\bar{u}, \bar{w}) = \sum_i |u_i - w_i|$.

Least-squares regression



$$\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} - \mu_x \begin{pmatrix} 1\\ 1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix} x_1-\mu_x\\ \vdots\\ x_n-\mu_x\end{pmatrix}$$

The closest point to

$$\begin{pmatrix} y_1\\ \vdots\\ y_n\end{pmatrix}$$

is

$$\left[\frac{1}{n\,Var[X]}\begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix}\cdot\begin{pmatrix} x_1-\mu_x\\ \\ x_n-\mu_x\end{pmatrix}\right]\begin{pmatrix} x_1-\mu_x\\ \vdots\\ x_n-\mu_x\end{pmatrix} + \mu_Y\begin{pmatrix} 1\\ 1\\ 1\\ 1\end{pmatrix}$$

$$a = \frac{E[XY]-E[X]E[Y]}{Var[X]}, \qquad b = E[Y] - a\,E[X]$$

Fitting curves

$$\min \sum_{i=1}^{n} \delta_i$$

Subject to

$$x_i^2 a + x_i b + c - \delta_i \leq y_i$$

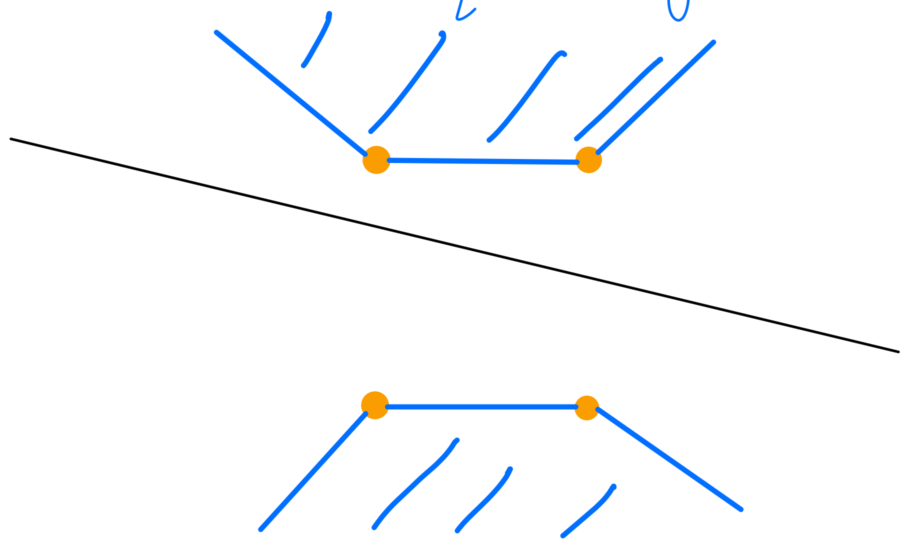
$$-x_i^2 a - x_i b - c - \delta_i \leq -y_i$$

... and so on for polynomials of higher degree

Uniqueness (Janaky)

Solutions are not unique in general.

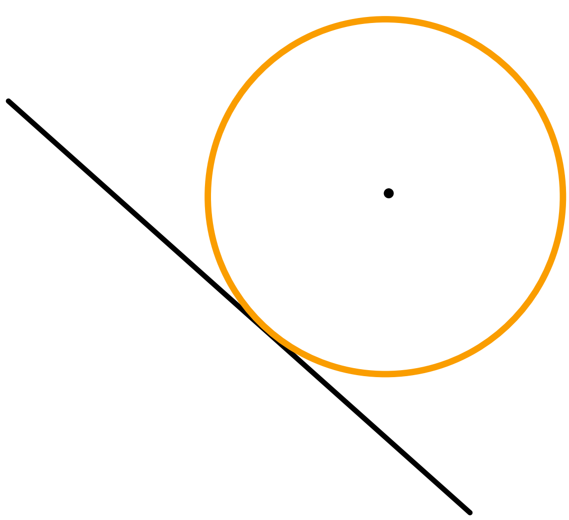
(Sreejata)

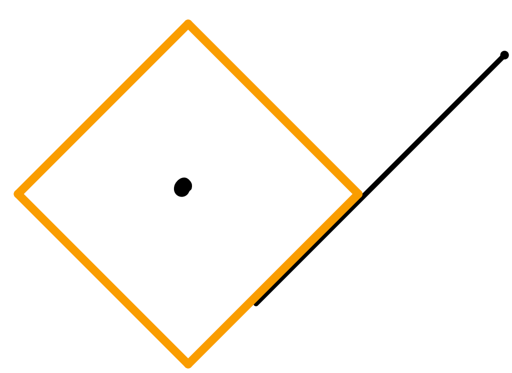


What does the set of optimal lines look like in general?

Why are there multiple solutions? (Sayantan)

ℓ_2 ℓ_1





5 separation of points

LP

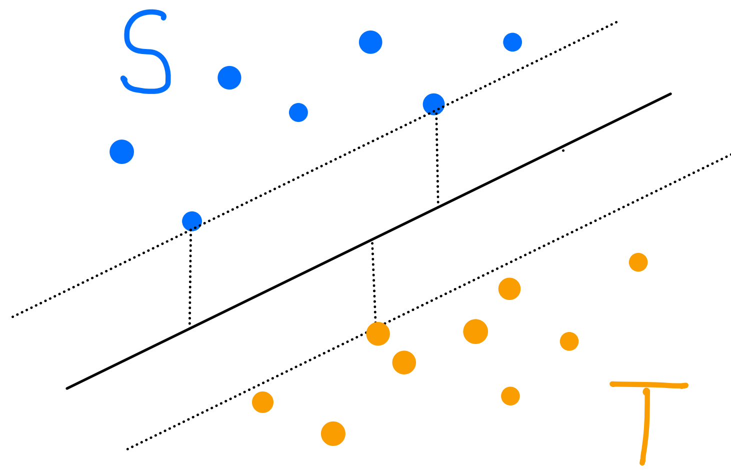
maximize δ

$x_i a + b + \delta \leq y_i$

for $i \in S$

$-x_i a - b + \delta \leq -y_i$

for $i \in T$



Three cases
S above, T below
T below, T above
Separation by a vertical line

Similar LPs can be written for separation by polynomials of higher degree.

Multiple optimal solutions are rarer.

6 Largest disk in a convex polygon

A polygon is specified by n inequalities of the form

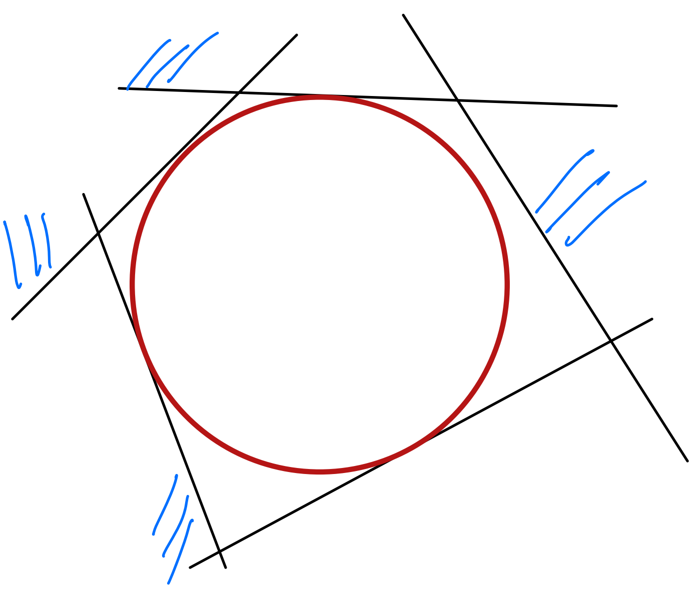
$$
\begin{aligned}
a_1^T x &\leq b_1 \\
a_2^T x &\leq b_2 \\
\vdots \quad & \quad \vdots \quad \vdots \\
a_n^T x &\leq b_n
\end{aligned}
$$

What is the largest disk that can be placed inside the polygon?

(a_i are unit vectors)

LP

$$
\begin{aligned}
\max \quad & r \\
\text{subject to} \quad & a_1^T x + r \leq b_1 \\
& a_2^T x + r \leq b_2 \\
& a_n^T x + r \leq b_n
\end{aligned}
$$

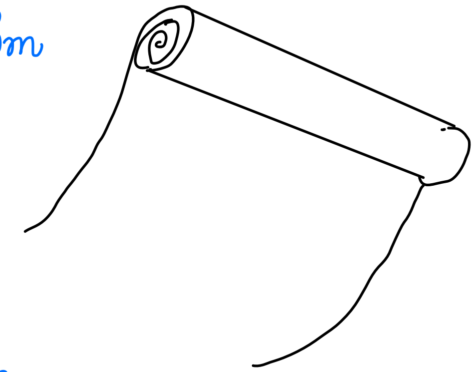


Can this be generalized to higher dimensions?

What if we wish to place an L1 disk inside a polyhedron?

7 Cutting paper rolls

Standard width of rolls = 3m



Order

97 rolls of width 135 cm

610 rolls of width 108 cm

395 rolls of width 93 cm

211 rolls of width 42 cm

Question: What is the smallest number of 3m rolls that have to be cut to meet the order?

$$
P = \begin{array}{c}
135 \\ 108 \\ 93 \\ 42
\end{array}
\begin{bmatrix}
2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\
0 & 1 & 1 & 3 & 2 & 0 & 2 & 4 & 0 & 2 & 4 & 7
\end{bmatrix}
$$

$P_1 \quad P_2 \quad P_3 \quad \bullet \quad \bullet \quad \ldots \quad P_{12}$

There are 12 patterns for cutting each 3m roll.

The LP

minimize $x_1 + x_2 + \cdots + x_{12}$

$$P \begin{pmatrix} x_1 \\ \vdots \\ x_{12} \end{pmatrix} \geq \begin{pmatrix} b \end{pmatrix}$$

$x_1, \ldots, x_{12} \geq 0$

Requirements

$$b = \begin{bmatrix} 97 \\ 610 \\ 395 \\ 211 \end{bmatrix}$$

$x_1 = 48.5$

$x_5 = 206.25$

$x_6 = 197.5$

Total 452.25

If integral solution is required, we could round up.

$x_1 = 49$
$x_5 = 207$
$x_6 = 198$
Total: 454

Actually,

$x_1 = 49$
$x_5 = 207$
$x_6 = 196$
$x_9 = 1$
Total: 453

A better integral solution.

Linear objective function

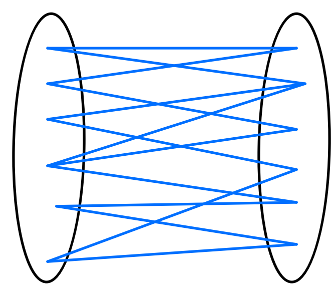
Linear constraints

But the variables are allowed to take only integer values.

Examples

$x_e \in [0,1]$

1 Bipartite matching



Input: A weighted bipartite graph.
$G = (V, E)$
$(w_e : e \in E)$

Output: A perfect matching of minimum weight

$$\min \sum_e w_e x_e$$

$$\sum_{e:\, v \in e} x_e = 1 \qquad \forall v \in V$$

$$0 \leq x_e \leq 1 \qquad \forall e \in E$$

$$x_e \in \mathbb{Z} \longleftarrow \text{integrality}$$

LP relaxation

We can drop the integrality constraint and solve the resulting LP.

- The fractional solution can be rounded to obtain an integral solution of the same cost!
- In fact, if the LP is solved using the simplex method, the solution will already be integral.

2 Vertex cover

Input: A graph $G = (V, E)$

Output: A subset $S \subseteq V$ s.t. every edge has at least one end point in S.

$$\begin{aligned} &\text{minimize} && \sum_{v \in V} x_v \\ &\text{subject to} && x_u + x_v \geq 1 \quad \text{for } \{u,v\} \in E \\ & && x_v \in \{0,1\} \quad \text{for } v \in V \end{aligned}$$

Integer Program

minimize $\sum_{v \in V} x_v$ LP relaxation

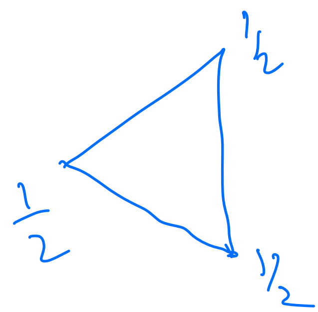
subject to $x_u + x_v \geq 1$

$0 \leq x_v \leq 1$

- Solve the LP to obtain a fractional solution $(x_v^* : v \in V)$

- $S_{LP} = \left\{ v : x_v^* \geq 1/2 \right\}$ is a feasible solution to the problem

- $|S_{LP}| \leq 2 |S_{OPT}|$



The LP relaxation yields a reasonable approximate solution.

Exercise: Consider the greedy algorithm that repeatedly picks a vertex that covers the maximum possible number of yet uncovered edges.

For each constant C (e.g., 10) construct a graph G_C for which the above algorithm yields a solution that is a factor C worse than the optimum.

3 Maximum independent set

Input: An undirected graph G

Output: A subset $I \subseteq V$ s.t. no two vertices in I are connected by an edge.

The integer program

$$\text{maximize} \quad \sum_{v \in V} x_v$$

$$\text{subject to} \quad x_u + x_v \leq 1 \quad \text{for each edge } \{u,v\} \in E,$$
$$x_v \in \{0,1\} \quad \text{for all } v \in V.$$

The LP relaxation

$$\text{maximize} \quad \sum_{v \in V} x_v$$

$$\text{subject to} \quad x_u + x_v \leq 1 \quad \text{for each edge } \{u,v\} \in E,$$
$$0 \leq x_v \leq 1 \quad \text{for all } v \in V.$$

The point

The lp solution is not even close.