Lecture 20: Copositive Programming

We have seen two important classes of cone programs so far: linear programs and semidefinite programs.

Both these classes are "easy" in the sense that there are efficient algorithms to solve them (for almost-optimal solutions in the semidefinite case).

We cannot expect similar results for general cone programs since the involved closed convex cones may be "hard". We will now see two concrete hard cones in SYM_n.

Definition. A matrix $M \in SYM_n$ is called copositive if $x^T M x \geq 0$ for all $x \in \mathbb{R}^n_+$.

Every positive semidefinite matrix is also copositive but the converse is not true. Consider $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is copositive (since all its entries are non-negative) but ~~it~~ is not psd since -1 is an eigenvalue.

Let $COP_n = \{M \in SYM_n : M \text{ is copositive}\}$ be the set of copositive matrices in SYM_n.

It is easy to show that COP_n is a closed convex cone. What is its dual cone COP_n^*?

Recall that for a cone $K \subseteq SYM_n$, the dual cone is
$K^* = \{Y \in SYM_n : Y \bullet X \geq 0 \text{ for all } X \in K\}$.

For any $x \in \mathbb{R}^n$, we have $x^T M x = M \bullet x x^T$, so a copositive matrix M satisfies $Y \bullet M \geq 0$ for any matrix

$Y = xx^T$. More generally, if $Y = \sum_{i=1}^{\ell} x_i x_i^T$ where $x_i \in \mathbb{R}^n_+$ for each i, then $Y \bullet M \geq 0$. So such matrices Y are certainly in COP_n^*. Let us give a name to such matrices.

<u>Definition.</u> A matrix $Y \in SYM_n$ is called <u>completely positive</u> if there exist vectors $\underbrace{x_1, x_2, \ldots, x_\ell}_{\text{(for some } \ell)}$ in $\mathbb{R}^n_+$ such that $Y = \sum_{i=1}^{\ell} x_i x_i^T$.

Note that $Y = AA^T$ where $A \in \mathbb{R}^{n \times \ell}$ with columns $x_1, x_2, \ldots, x_\ell$.

Every completely positive matrix is positive semidefn.
Indeed, M is psd $\Longleftrightarrow M = \sum_{i=1}^{\ell} x_i x_i^T$ for $x_1, \ldots, x_\ell$ in $\mathbb{R}^n$.

<u>Claim.</u> M is completely positive $\Longleftrightarrow$ there are $\binom{n+1}{2}$ vectors $x_1, \ldots, x_{\binom{n+1}{2}} \in \mathbb{R}^n$ s.t. $M = \sum_{i=1}^{\binom{n+1}{2}} x_i x_i^T$.

That is, though ℓ was arbitrary in the definition of completely positive matrices, we do not need to consider $\ell > \binom{n+1}{2}$. (Please prove this as an exercise.)

Let $POS_n = \{ M \in SYM_n : M \text{ is completely positive} \}$.
We have already seen that $POS_n \subseteq PSD_n \subseteq COP_n$.
We will next show that POS_n is a closed convex cone and $POS_n^* = COP_n$. Thus $COP_n^* = POS_n$.

<u>Lemma.</u> The set POS_n is a closed convex cone.

Proof. Observe that if $M = \sum_{i=1}^{\ell} x_i x_i^T \in POS_n$, then $\boxed{\lambda M = \sum_{i=1}^{\ell} (\sqrt{\lambda}\, x_i)(\sqrt{\lambda}\, x_i)^T}$ is also in POS_n for $\lambda \geq 0$.

For $M, M' \in POS_n$, we have $M = \sum_{i=1}^{\ell} x_i x_i^T$ and $M' = \sum_{i=1}^{\ell'} y_i y_i^T$ for some vectors $x_1, \dots, x_\ell, y_1, \dots y_{\ell'}$ in $\mathbb{R}^n_+$. Thus $M + M' = \sum_{i=1}^{\ell+\ell'} z_i z_i^T$ where

$z_i = x_i$ for $1 \leq i \leq \ell$ and $z_{\ell+i} = y_i$ for $1 \leq i \leq \ell'$.

So $M+M' \in POS_n$.

It remains to prove that the cone POS_n is closed.

Let $M_1, M_2, \dots$ be a sequence of completely positive matrices such that $\lim_{k\to\infty} M_k = M$. We will show that M is also completely positive. This will imply that the cone POS_n contains all its limit points, in other words, the cone POS_n is closed.

We have $M_k = A_k \cdot A_k^T$ for some non-negative matrix A_k, for each k.

Let $a_k[i]$ denote the i-th row of A_k, for $i = 1, \dots, n$.

Thus
$$M[i,i] = \lim_{k\to\infty} M_k[i,i] = \lim_{k\to\infty} a_k[i] \cdot a_k[i]^T = \lim_{k\to\infty} \|a_k[i]\|^2,$$
for $i = 1, \dots, n$.

This implies that the sequence of vectors $(a_k[i])_{k\in N}$ is bounded. Hence there is a convergent subsequence (with limit $a[i]$). This yields $a[i] \in \mathbb{R}^n_+$ and $M[i,i] = \|a[i]\|^2$. By a similar argument,
$$M[i,j] = \lim_{k\to\infty} a_k[i] \cdot a_k[j]^T.$$

So $M = AA^T$ where A is the non-negative matrix with rows $a[1], a[2], \ldots, a[n]$. Hence $M \in POS_n$. ▨

Theorem. $POS_n^* = COP_n$.

Proof. Let $M \in COP_n$. We will now show that $M \cdot X \geq 0$ for all $X \in POS_n$. This will imply that $M \in POS_n^*$, thus $COP_n \subseteq POS_n^*$.

We have
$$\underbrace{M}_{\in COP_n} \cdot \underbrace{\sum_{i=1}^{l} x_i x_i^T}_{\substack{\in POS_n \\ (\text{so } x_i \in \mathbb{R}_+^n \ \forall i)}} = \sum_{i=1}^{l} M \cdot x_i x_i^T$$
$$= \sum_{i=1}^{l} x_i^T M x_i \geq 0.$$
(since $x_i \in \mathbb{R}_+^n \ \forall i$ and $M \in COP_n$)

We now need to show that $COP_n \supseteq POS_n^*$. Consider $M \notin COP_n$. Then there exists $\tilde{x} \in \mathbb{R}_+^n$ such that $\tilde{x}^T M \tilde{x} < 0$. This means that $M \cdot \tilde{x}\tilde{x}^T < 0$ where $\tilde{x} \in \mathbb{R}_+^n$. So $M \notin POS_n^*$.

Thus $M \notin COP_n \Rightarrow M \notin POS_n^*$. Hence $POS_n^* \subseteq COP_n$.

That is, $POS_n^* = COP_n$. ▨

Reference

- Approximation Algorithms and SDP. (Chapter 7)
  – B. Gärtner and J. Matoušek.