Lecture 27: The ARV Algorithm

Recall our initial assumption that $|B(i, 1/4)| < \frac{n}{4}$ for all $i \in V$. This assumption was justified by showing that if there was some $i \in V$ with $|\{j \in V : d(i,j) \leq 1/4| \geq n/4$ then we could efficiently find $S \subseteq V$ with $\rho(S) = O(1) \cdot \rho(G)$.

Our next claim was the following. (This is slightly different from what we did in class.)

Claim. There exists $\theta \in V$ s.t. $|B(\theta, 8)| \geq \frac{3n}{4}$. Furthermore, let $\mathcal{U} = B(\theta, 8) - B(\theta, 1/4)$. Then $|\mathcal{U}| \geq n/2$ and for all $i \in \mathcal{U}$, there exist at least $n/4$ vertices $j \in \mathcal{U}$ s.t. $d(i,j) > \frac{1}{4}$.

Proof. Suppose no such θ exists. Then for all $i \in V$, we have more than $n/4$ vertices at distance > 8. So we have

$$\underbrace{\sum_{i,j \in V} d(i,j)}_{\text{This is } 2n^2.} = \sum_{i \in V}\left(\sum_{j \in V} d(i,j)\right) > n \cdot 8\frac{n}{4} = 2n^2,$$

a contradiction. Thus there exists some $\theta \in V$ s.t. $|B(\theta, 8)| \geq 3n/4$. Since $|B(\theta, 1/4)| \leq n/4$ by our initial assumption, for $\mathcal{U} = B(\theta, 8) - B(\theta, \frac{1}{4})$

$$|\mathcal{U}| \geq |B(\theta, 8)| - |B(\theta, 1/4)| \geq \frac{3n}{4} - \frac{n}{4} = \frac{n}{2}.$$

Finally pick any $i \in \mathcal{U}$. Since $|B(i, 1/4)| < n/4$ while $|\mathcal{U}| \geq n/2$, there are $\geq \frac{n}{4}$ $j \in \mathcal{U}$ s.t. $d(i,j) > 1/4$. ▨

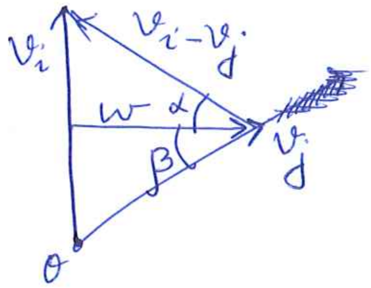
Recall the sets $\mathcal{L}$ and R computed in the algorithm.

Theorem. There is a constant $c > 0$ such that $\Pr\left[|\mathcal{L}| \geq cn \text{ and } |R| \geq cn \right] \geq c$.

Proof. Let us pick $i, j \in \mathcal{U}$ such that $d(i,j) > \frac{1}{4}$. We have $\frac{1}{4} \leq \|v_i\|^2 \leq 8$ (recall that we moved the origin to θ) and so $\frac{1}{4} \leq \|v_j\|^2 \leq 8$.

Without loss of generality, assume $\|v_i\| \geq \|v_j\|$. Let

$$w = v_j - (v_j, v_i) \cdot \frac{v_i}{\|v_i\|^2}.$$



We have the following inequalities:

$$\|v_i\|^2 \leq \|v_i - v_j\|^2 + \|v_j\|^2$$
$$\|v_i - v_j\|^2 \leq \|v_i\|^2 + \|v_j\|^2.$$

The first inequality says that the angle between v_j and $v_i - v_j$ is not obtuse and the second says that the angle between v_i and v_j is not obtuse.

Let α and β be as indicated in the picture above. Then $\alpha + \beta \leq \pi/2$.

- if $\alpha \leq \pi/4$ then $\|w\| = \|v_i - v_j\| \cos\alpha$
$$\geq \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$$
(since $d(i,j) = \|v_i - v_j\|^2 > \frac{1}{4}$).
- if $\beta \leq \pi/4$ then $\|w\| = \|v_j\| \cos\beta$
$$\geq \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$$
(since $\|v_j\|^2 \geq 1/4$)

Thus $\|w\| \geq \frac{1}{2\sqrt{2}}$.

Recall that we need to bound $Pr[i \in \mathcal{L}$ and $j \in R]$

<u>Claim.</u> If $r^T v_i \in [-2, -1]$ and $r^T w \geq 3$ then $i \in \mathcal{L}$ and $j \in R$.

<u>Proof.</u> We know that $r^T v_i \leq -1$ puts $v_i \in \mathcal{L}$. What we need to show now is that $r^T v_j \geq 1$. That will put $j \in R$.

Since $v_j = (v_i, v_j) \frac{v_i}{\|v_i\|^2} + w$, taking a dot product with r on both sides,

$$(r, v_j) = (r, v_i) \frac{(v_i, v_j)}{\|v_i\|^2} + (r, w)$$

$$\geq -2 \cdot \underset{\sim}{1} + 3 = 1.$$

$$(\text{since } \|v_j\| \leq \|v_i\|)$$

So we have $Pr[i \in \mathcal{L}$ and $j \in R] \geq Pr[-2 \leq r^T v_i \leq -1$ and $r^T w \geq 3]$.

This equals (since <u>v_i & w are orthogonal</u>)

$$Pr\left[\frac{-2}{\|v_i\|} \leq \frac{r^T v_i}{\|v_i\|} \leq \frac{-1}{\|v_i\|}\right] \cdot Pr\left[\frac{r^T w}{\|w\|} \geq \frac{3}{\|w\|}\right].$$

Here we used the fact that $r^T x$ and $r^T y$ are independently distributed for vectors x and y orthogonal to each other.

For any unit vector x, $r^T x \sim N(0,1)$. The prob. that a random variable $\sim N(0,1)$ takes a value in an $\Omega(1)$-sized interval is at least a constant.

Thus $\Pr\left[\frac{-2}{\|v_i\|} \leq \frac{r^T v_i}{\|v_i\|} \leq \frac{-1}{\|v_i\|}\right] = \Omega(1)$ and similarly,

$\Pr\left[\frac{r.w}{\|w\|} \geq \frac{3}{\|w\|}\right] = \Omega(1)$.

Recall that $|\mathcal{U}| \geq n/2$. Also each $i \in \mathcal{U}$ has $\geq n/4$ vertices $j \in \mathcal{U}$ s.t. $d(i,j) \geq 1/4$. Thus $E[|L_{\mathcal{U}} \times R_{\mathcal{U}}|]$ is $\Omega(n^2)$, where $L_{\mathcal{U}} = L \cap \mathcal{U}$ and $R_{\mathcal{U}} = R \cap \mathcal{U}$. This leads to the theorem statement. ▨

A high level overview of the structure theorem. (by Barak & Steurer, 2016)

We know that $\frac{r^T v}{\|v\|} \sim N(0,1)$. This implies that

$$\Pr[r^T v \geq \alpha] \leq e^{-\alpha^2/\|v\|^2}.$$

Thus $\Pr[r^T(v_i - v_j) \geq C\sqrt{\ln n}] \leq e^{\frac{-C^2 \ln n}{\|v_i - v_j\|^2}}$

Since $\|v_i - v_j\|^2 \leq \|v_i\|^2 + \|v_j\|^2 \leq 16$, this probability is at most $e^{-\frac{C^2 \ln n}{16}} = \frac{1}{n^{C^2/16}}$. This constraint holds for all $i,j \in \mathcal{U}$.

So for sufficiently large C, we have $r^T(v_i - v_j) \leq C\sqrt{\ln n}$ for all $i,j \in \mathcal{U}$ whp. Then one can show that

$$E\left[\max_{i,j \in \mathcal{U}} r^T(v_i - v_j)\right] \leq C\sqrt{\ln n}.$$

Projection lemma. $\frac{1}{\Delta}\left(\frac{E[|M|]}{n}\right)^3 \leq E\left[\max_{i,j \in \mathcal{U}} r^T(v_i - v_j)\right]$

$\leq C\sqrt{\ln n}$.

From the projection lemma, for the right choice of constants and $\Delta = \Omega\left(\frac{1}{\sqrt{\log n}}\right)$, we get $\left(\frac{E[|M|]}{n}\right)^3 \leq \left(\frac{C}{2}\right)^6$, which proves the structure theorem.

So the projection lemma implies the structure theorem.