Duality

The original LP

$$\text{maximize} \quad c^T x$$
$$\text{subject to} \quad Ax = b$$
$$x \geq 0$$

Idea: Obtain new equations by taking linear combinations of existing equations.

A typical such equation looks like

$$(y_1\ y_2 \ldots y_m)\ A x = (y_1\ y_2 \ldots y_m)\ b$$

i.e. $$y^T A x = y^T b$$

Suppose we choose y so that

$$c^T = y^T A,$$

then

$$c^T x = y^T A x = y^T b,$$

that is, $$y^T b = Opt$$

We can say more since $x \geq 0$.

Suppose we can arrange y^T such that

$$c^T \leq y^T A,$$

then

$$c^T x \leq y^T A x = y^T b$$

To obtain the best upper bound, solve the LP.

$$\boxed{\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c \\ & y \in \mathbb{R}^m \end{array}}$$

DUAL

~~non-negativity~~

Then, if x is feasible for the PRIMAL and y is feasible for the DUAL, then

$$\underbrace{\text{obj}_{\text{PRIMAL}}(x)}_{\text{maximization}} = c^T x \leq y^T A x = y^T b = \underbrace{\text{obj}_{\text{DUAL}}(y)}_{\text{minimization}}$$

Questions: Is the DUAL feasible?

Is the DUAL unbounded?

Theorem: If both are bounded, then

$$\text{Opt}(\text{PRIMAL}) \leq \text{Opt}(\text{DUAL}).$$

When do we have equality?

$\boxed{\text{SIMPLEX} \Rightarrow \text{ALWAYS.}}$

Why?

$$\max \quad c^T x$$
$$\text{subject to} \quad Ax = b$$
$$x \geq 0$$

$$\begin{array}{l} x_B = A_B^{-1} b - A_B^{-1} A_N x_N \\ \hline z = c^T x = L - r_0 x_N \end{array} \xrightarrow{\text{SIMPLEX}} \begin{array}{l} x_{B_f} = A_{B_f}^{-1} b - A_{B_f}^{-1} A_{N_f} x_{N_f} \\ \hline z = \text{OPT} - r x_{N_f} \end{array}$$

$r \geq 0$

final tab

Focus on the last row: $\exists y$

$$c^T x + y^T (b - Ax) = \text{OPT} - r x_{N_f}$$
$$\underbrace{c^T x}_{c_j} + \underline{y^T b} = \underline{\text{OPT}} + \underbrace{y^T A x - r x_{N_f}}_{(y^T A)_j - r_j}$$

Rules for dualization

	PRIMAL	DUAL
variables	$x_1, x_2, \dots, x_n$	$y_1, y_2, \dots, y_m$ (multipliers for the rows)
matrix	$A \in \mathbb{R}^{m \times n}$	$A^t \in \mathbb{R}^{n \times m}$
RHS	$b \in \mathbb{R}^m$	$c \in \mathbb{R}^n$
Objective function	max $c^T x$	min $b^T y$
Constraints	$\leq$	$y_i \geq 0$
	$\geq$ m rows	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	$\geq$
	$x_j \leq 0$	$\leq$ n rows
	$x_j \in \mathbb{R}$	$=$

PRIMAL \ DUAL	INFEASIBLE	FEASIBLE BOUNDED	FEASIBLE UNBOUNDED
INFEASIBLE	✓	✗	✓
FEASIBLE BOUNDED	✗	✓	✗
FEASIBLE UNBOUNDED	✓	✗	✗

$c^T x^* = b^T y^*$

Farkas lemma

$$y^T \underline{Ax} \geqslant 0$$
$$y^T b \quad = 0$$

(i) $\begin{array}{l} Ax = b \\ x \geqslant 0 \end{array}$ is feasible $\Leftrightarrow$ $\begin{array}{l} y^T A \geqslant 0^T \\ y^T b = -1 \end{array}$ is infeasible.

(ii) $\begin{array}{l} Ax \leqslant b \\ x \geqslant 0 \end{array}$ is feasible $\Leftrightarrow$ $\begin{array}{l} y^T A \geqslant 0^T \\ y^T b = -1 \\ y \geqslant 0 \end{array}$ is infeasible.

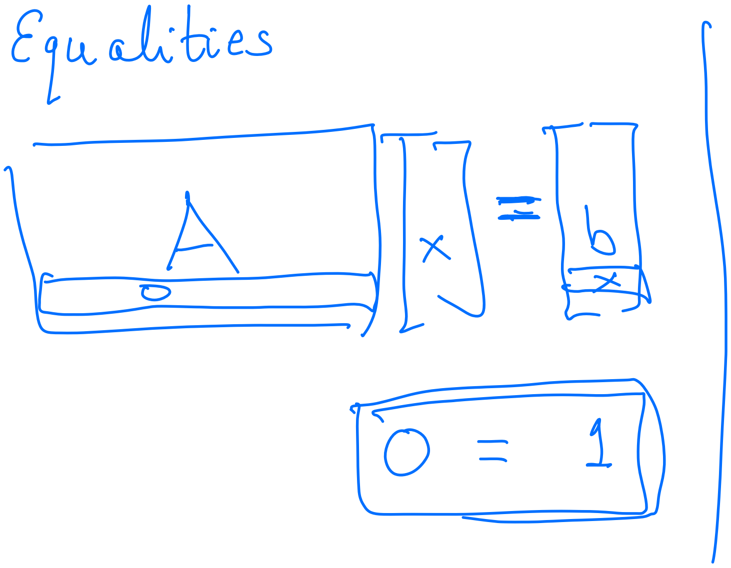
(iii) $Ax \leqslant b$ is feasible $\Leftrightarrow$ $\begin{array}{l} y^T A = 0^T \\ y^T b \geqslant 0 \end{array}$ is infeasible.

Principle

If a system of inequalities $Ax \leqslant b$ is infeasible, then it is possible to derive the contradiction $0 \leqslant -1$ from them by adding together non-negative multiples of the inequalities.

The Fourier - Motzkin elimination method

Equalities



Finding a solution to $\underline{Ax \leq b}$ by variable elimination.

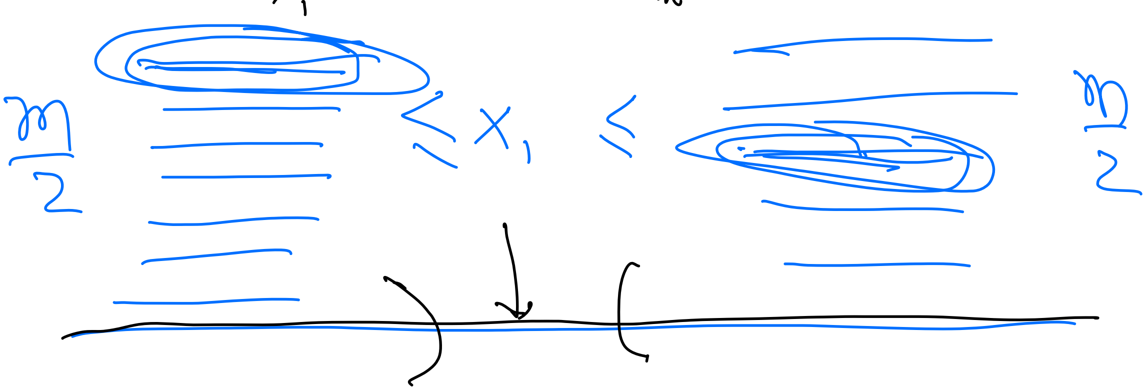
Consider the first variable

$5x_1 \quad - - - - \quad \leq b_1$ upper bound on x_1

$-7x_1 \quad . . . . \quad \leq b_2$ lower bound on x_1

$x_1 \quad - \quad .$

$x_1 \quad . \quad . \quad b_m$



Fourier – Motzkin

$$\boxed{\checkmark} \le x_m \le \boxed{\checkmark}$$

$$0 \le -27$$

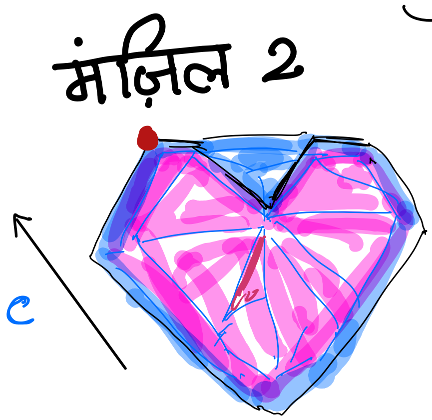
$$\boxed{0 \le -1}$$

Eliminate x_1 by joining each lower bound with each upper bound. Include the inequalities that do not involve x_1.

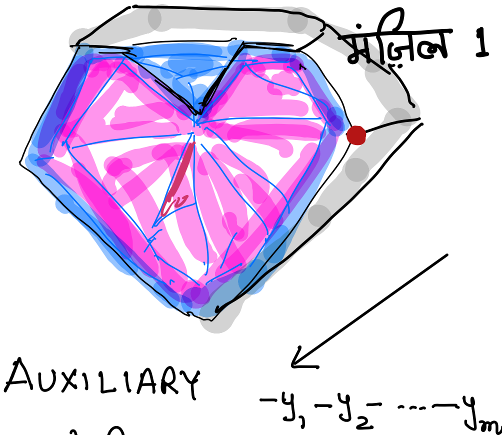
Since the original system was infeasible, we must end with

$$0 \le -1.$$

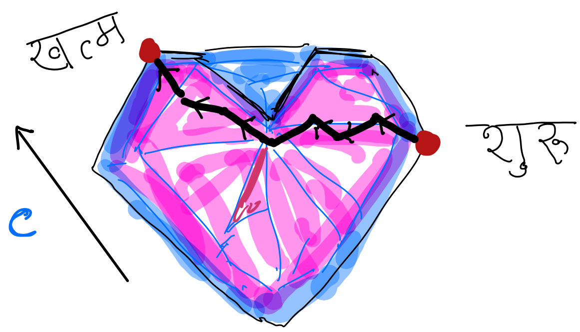
SUMMARY



ORIGINAL LP



AUXILIARY LP



दिल अपना और प्रीत पराई

अजीब दास्तां है ये

कहाँ शुरू कहाँ ख़तम

ये मंज़िलें हैं कौन सी

न वो समझ सके न हम

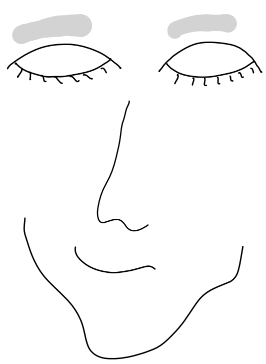
Lexicographic rule
Bland's rule

~~cycling~~

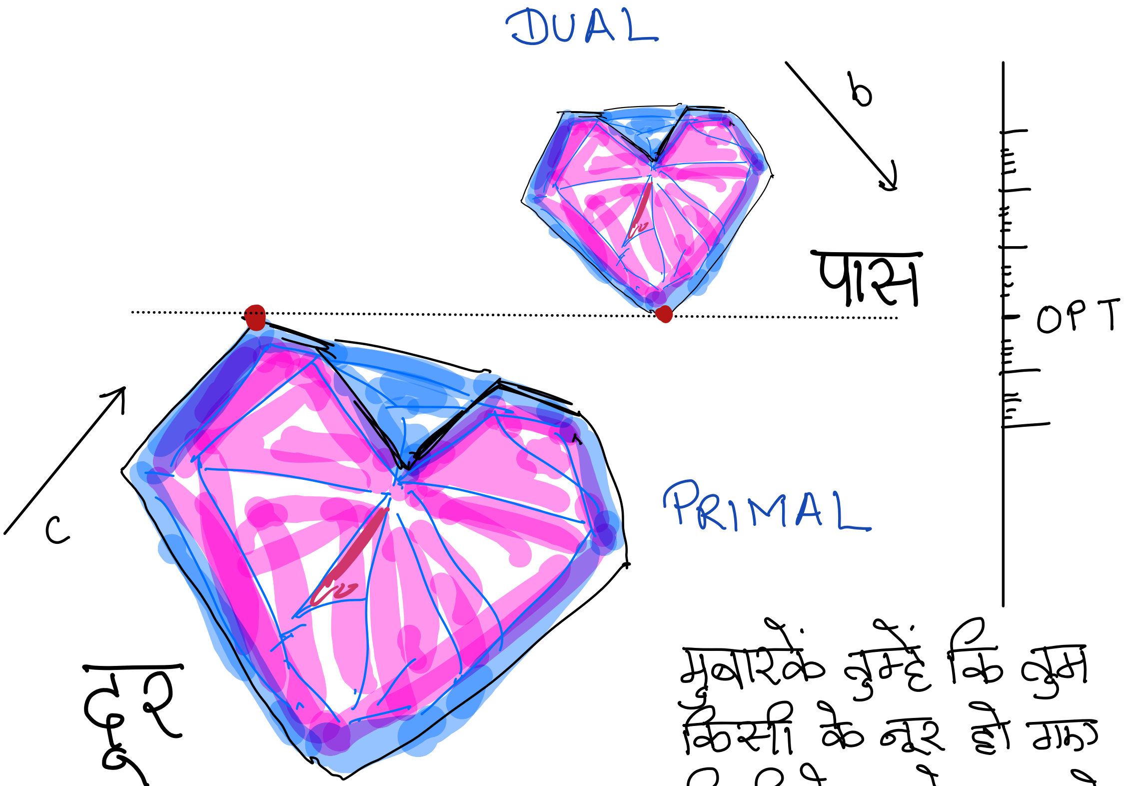
Degeneracy

Klee-Minty
examples

ये रोशनी के साथ क्यों
धुआं उठा चिराग से
ये ख़्वाब देखती हूँ मैं
कि जग पड़ी हूँ ख़्वाब से

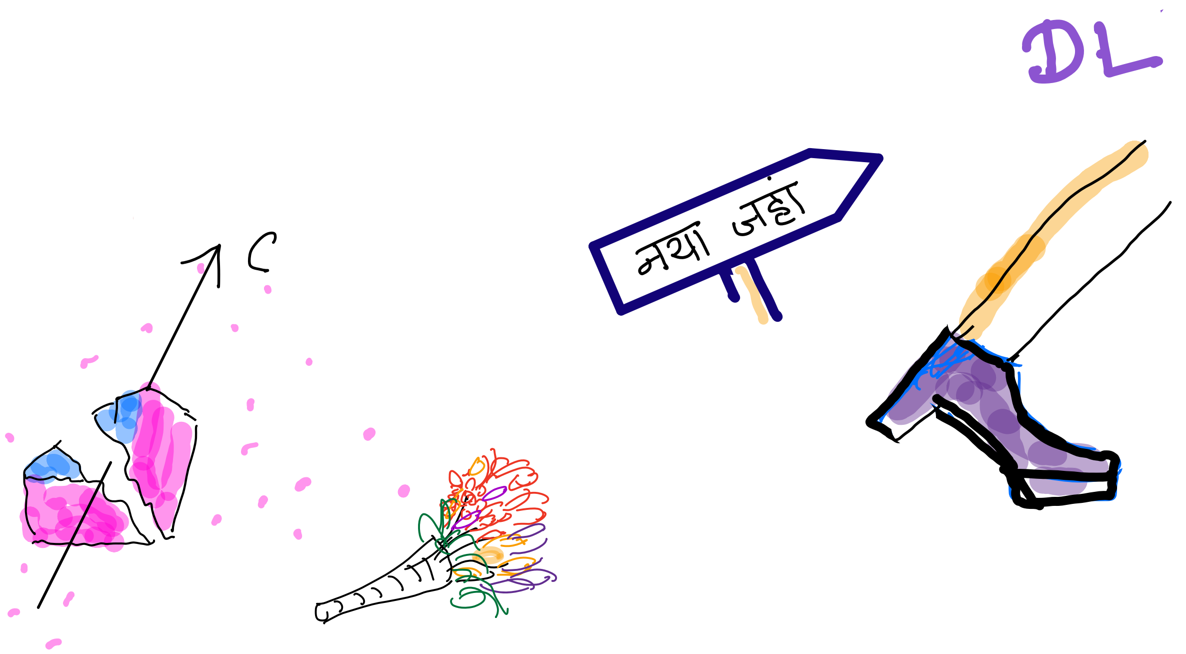


Randomized pivo-ting
Hirsch Conjecture
Smoothed analysis
Average case
Efficient in practice



मुबारकें तुम्हें कि तुम
किसी के नूर हो गए
किसी के इतने पास हो
कि सबसे दूर हो गए

किसी का प्यार लेके तुम
नया जहां बसाओगे
ये शाम जब भी आएगी
तुम हमको याद आओगे



ये शाम जब भी आएगी तुम हमको याद आओगे

अजीब दास्तां है ये

कहाँ शुरू कहाँ ख़तम

ये मंज़िलें हैं कौन सी

न वो समझ सके न हम

Music

https://m.youtube.com/watch?v=PmgVX-0W3vk