Lecture 21 Online Paging

We were analyzing the Marker algorithm. We now want to bound the expected number of misses by Marker algorithm.

- Consider round t. Call a page **old** if it was requested in round $t-1$, and **new** otherwise.
- So all old pages are in Marker's cache at the beginning of the round, i.e, they are in S_M.
- Let there be l_t many new pages in round t.
- Recall that the cache size is k. So there are $k-l_t$ many old pages in round t.

Observe that every single new page request will cause a miss by Marker algorithm. That is, the first time this page is requested in round t, it will be a miss.

As for the old pages, the first time an old page is called in round t → it will cause a miss by Marker algorithm only if this page got evicted previously by the algorithm in this round.

In order to maximize the number of misses, let us assume that the l_t requests for new pages precede the $k-l_t$ requests for old pages.

* These l_t page requests are certainly misses.
What is the probability that the first of the old page requests is a miss?

- It is easy to see that this is $\frac{l_t}{k}$.

What is the probability that the 2nd of the old page requests is a miss? That is, this is the first time this old page is requested in this round and it is the second ~~[illegible]~~ old page to be requested in this round, what is the probability that this page is not present in the cache?

* We claim this probability is $\frac{l_t}{k-1}$.

This is because l_t pages out of $k-1$ old pages have been evicted so far. This is by the following reasoning:

- the first old page was in the cache at the beginning of the round and it must be there now (when the second old page was requested).
- it may have been evicted but it has been brought back into the cache by evicting another old page.
- thus l_t pages have been evicted out of $k-1$ old pages.

Hence Prob(the 2nd of the old page requests is a miss) $= l_t/(k-1)$.

It is easy to see that Prob(the i-th of the old page requests is a miss) $= l_t/(k-i+1)$.

We mimic the above argument to show this. We claim l_t pages out of $k-i+1$ old pages have been evicted so far.

- Even if some of the first $(i-1)$ old pages have been evicted, they have been brought back by evicting other old pages.
- Thus l_t pages have been evicted out of $(k-i+1)$ old pages.

Hence $E[f_{\text{Marker}}(r_1, \ldots, r_N)] \leq \sum_{\text{rounds}}$ Expected number of misses in this round

Expected number of misses in the t-th round

$$\leq l_t + \frac{l_t}{k} + \frac{l_t}{k-1} + \cdots + \frac{l_t}{l_t+1}$$

$$\leq l_t \cdot H_k$$

Thus $E[f_{\text{Marker}}(r_1, \ldots, r_N)] \leq \sum_t l_t \cdot H_k$

Hence competitive ratio of the Marker algo.

$$\leq \frac{\sum_t l_t \cdot H_k}{\sum_t l_t / 2} = 2H_k.$$

Thus deterministic algorithms for online paging have competitive ratio $\geq k$ while the above randomized algorithm has competitive ratio $\simeq 2\ln k$.

Linear Programming

Linear programming is the problem of minimizing or maximizing a linear function subject to linear constraints. The function being optimized is called the objective function.

Ex. min $3x_1 + 4x_2$

subject to

$$\left.\begin{aligned} x_1 + 2x_2 &\geq 5 \\ x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}\right\}$$

Any solution that satisfies all the constraints is called a feasible solution.

In the above problem, the optimal solution is $x_1 = 3$, $x_2 = 1$. So the optimal value of this LP is $3 \times 3 + 4 \times 1 = 13$.

A feasible solution that minimizes the objective function is called an optimal solution.

Is there any way we can quickly convince someone that 13 is the best answer here?

$$1\cdot(x_1 + 2x_2) + 2\cdot(x_1 + x_2) \geq 1\times 5 + 2\times 4 = 13.$$

Thus $3x_1 + 4x_2 \geq 13$.

Is it always possible to find such multipliers such that

$$y_1(\text{Inequality } 1) + \ldots + y_n(\text{Inequality } n) = \sum_{i=1}^{n} b_i y_i = \sum_{j=1}^{k} c_j x_j$$

Let us look at another example.

$$\min\ 7x_1 + 3x_2$$

such that
$$-x_1 + x_2 \geq 10$$
$$2x_1 + 5x_2 \geq 15$$
$$x_1, x_2 \geq 0$$

Let y_1 be the multiplier for Inequality 1 & y_2 for Inequality 2.

Let λ_1, λ_2 be the multipliers for $x_1 \geq 0$, $x_2 \geq 0$.

Is there a setting of $\underbrace{y_1, y_2, \lambda_1, \lambda_2}_{\text{non-negative}}$ so that

$$y_1(-x_1 + x_2) + y_2(2x_1 + 5x_2) + \lambda_1 x_1 + \lambda_2 x_2 = 7x_1 + 3x_2$$

Then we know that $7x_1 + 3x_2 \geq 10y_1 + 15y_2$. So the optimal value is bounded from below by $10y_1 + 15y_2$. We want to find the best setting of y_1, y_2 to get as good a lower bound as possible. This problem of determining the best values for y_1, y_2 is the dual problem.

$$\max\ 10y_1 + 15y_2$$

s.t.
$$-y_1 + 2y_2 + \lambda_1 = 7$$
$$y_1 + 5y_2 + \lambda_2 = 3$$
$$y_1, y_2, \lambda_1, \lambda_2 \geq 0.$$

Another way of writing these constraints is

$$\begin{aligned} -y_1 + 2y_2 &\leq 7 \\ y_1 + 5y_2 &\leq 3 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Primal LP: let there be k variables & n inequalities.

$$\min \sum_{j=1}^{k} c_j x_j$$

s.t.

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \geq \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1, \ldots, x_k \geq 0$$

Dual LP

$$\max \sum_{i=1}^{n} b_i y_i$$

s.t. $$A^T \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \leq \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

$$y_1, \ldots, y_n \geq 0.$$

Weak duality theorem

If $(x_1, \ldots, x_k)$ is primal feasible and $(y_1, \ldots, y_n)$ is dual feasible then $\sum_j c_j x_j \geq \sum_i b_i y_i$

Proof.

Consider $(y_1 \ldots y_n) A \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$. This equals $\sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} x_j y_i$

This sum can be written as

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{k} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{n} b_i y_i$$

(since $(x_1, \ldots, x_k)$ is primal feasible)

$$\| $$

$$\sum_{j=1}^{k} \left(\sum_{i=1}^{n} a_{ij} y_i \right) x_j \leq \sum_{j=1}^{k} c_j x_j$$

(since $(y_1, \ldots, y_n)$ is dual feasible). □

An important question: Is there a gap between the primal optimal value and the dual optimal value?

Strong duality theorem. If $(x_1^*, \ldots, x_k^*)$ is the optimal solution for the primal LP then the dual LP also has an optimal soln. $(y_1^*, \ldots, y_n^*)$ and $\sum_j c_j x_j^* = \sum_i b_i y_i^*$.

There are 3 possibilities for the primal LP.
1) Infeasible 2) Unbounded 3) Finite optimal value.

Similarly there are 3 possibilities for the dual LP.
1') Infeasible 2') Unbounded 3') Finite optimal value.

1) $\Rightarrow$ optimal value of the primal LP is the minimum of the empty set $= \infty$.

Similarly 1') $\Rightarrow$ optimal value of the dual LP is the maximum of the empty set $= -\infty$.

2) $\Rightarrow$ primal opt $= -\infty$ and 2') $\Rightarrow$ dual opt $= \infty$.

Observe ~~that~~ 2) $\Rightarrow$ 1') by weak duality.
Similarly observe that 2') $\Rightarrow$ 1).

We can also have both 1) and 1') occurring together. Then there is ∞ gap between the primal optimal value and dual optimal value.

An example for this (both primal LP and dual LP are infeasible)

min $x_1 - 2x_2$	max $2y_1 - y_2$
s.t.	s.t.
$x_1 - x_2 \geq 2$	$y_1 - y_2 \leq 1$
$-x_1 + x_2 \geq -1$	$-y_1 + y_2 \leq -2$
$x_1, x_2 \geq 0$	$y_1, y_2 \geq 0$.

Thus there are 4 possibilities in total:
1) and 1'); 2) and 1'); 1) and 2');
$\underbrace{\text{3) and 3')}}_{\text{this is strong duality.}}$

$\boxed{\text{3)} \Rightarrow \text{3') and 3')} \Rightarrow \text{3)}}$