# Lecture 22 Linear Programming

In this lecture we will see a proof of strong duality theorem. We will model this as a 2-player game. The 2 players will be called the MIN player and the MAX player.

Game 1: The min player plays first. He chooses values for the variables that he controls.

The max player plays second. He chooses values for the variables that he controls.

Value of the game $= f(x_1, \dots, x_k, y_1, \dots, y_n, \lambda_1, \dots, \lambda_k)$ where $x_1, \dots, x_k$ are the variables that the min player controls and $y_1, \dots, y_n, \lambda_1, \dots, \lambda_k$ are the variables that the max player controls.

We want the value of this game to be $\boxed{OPT_{primal}}$ if the values $x_1^0, \dots, x_k^0$ chosen by the min player are primal feasible. Else we want the value of the game to be $\infty$.

How do we enforce this?

- The max player controls $n+k$ variables

$$\underbrace{y_1, \dots, y_n}_{\text{one for each constraint of the primal LP}} + \underbrace{\lambda_1, \dots, \lambda_k}_{\text{one variable for each constraint } x_i \geq 0}$$

all the $y_i$'s and $\lambda_j$'s have to be $\geq 0$

Let $f(x_1, \dots, x_k, y_1, \dots, y_n, \lambda_1, \dots, \lambda_k)$

$$= c_1 x_1 + \dots + c_k x_k + \sum_i y_i \left(b_i - \sum_j a_{ij} x_j\right) - \sum_j \lambda_j x_j$$

If the values $x_1^0, \dots, x_k^0$ chosen by the min player are primal feasible then the max player sets all $y_i$'s and $\lambda_j$'s to 0.

If some constraint of the primal LP is violated then the max-player makes the appropriate $y_i$ or $\lambda_j \to \infty$. This makes the value of the game $\to \infty$.

Claim 1. Value of Game 1 is $\boxed{OPT_{primal}}$

Proof. Value of Game 1 $\leq OPT_{primal}$ since every value attained by the primal LP can also be achieved in Game 1 by the min player taking the very same values of $x_1, \dots, x_k$. Thus by taking the optimal solution $(x_1^*, \dots, x_k^*)$ of the primal LP, the value of Game 1 becomes $OPT_{primal}$.

Suppose value of Game 1 $< OPT_{primal}$. Then some primal LP constraint got violated by the values $x_1^\circ, \dots, x_k^\circ$ chosen by the min player. However this makes the value of Game 1 $\to \infty$. But $OPT_{primal} < \infty$. This contradicts Game 1 $< OPT_{primal}$. □

Game 2. Now the max player plays first. He chooses $y_1, \dots, y_n, \lambda_1, \dots, \lambda_k$: all these values have to be $\geq 0$.

The min player plays second - he chooses $x_1, \dots, x_k$.

Value of the game $= f(x_1, \dots, x_k, y_1, \dots, y_n, \lambda_1, \dots, \lambda_k)$.

Claim 2. Value of Game 2 is $\boxed{OPT_{dual}}$

Proof. Observe that $f(x_1, \dots, x_k, y_1, \dots, y_n, \lambda_1, \dots \lambda_k)$ can also be written as $\sum_i b_i y_i +$

$$\sum_j x_j \left(c_j - \sum_i a_{ij} y_i - \lambda_j\right)$$

The min player plays after the max player and he is unconstrained — he can choose $x_1, \dots, x_k$ to be non-negative or negative (recall that we said the max player is constrained to choose only non-negative values for his variables).

— How can the max player make the min player totally irrelevant?

The only way the max player can make the min player totally irrelevant is by taking

$$\sum_i a_{ij} y_i + \lambda_j = c_j \text{ for each } j.$$

That is, $\sum_i a_{ij} y_i \leq c_j$ for $j = 1, \dots, k$.

Since $y_1, \dots, y_n$ anyway satisfy $y_i \geq 0 \ \forall i$, the max player's strategy is to choose $y_1, \dots, y_n$ to be dual feasible.

— If the max player violates any of the dual LP constraints then the min player makes the value of game 2 $\to -\infty$.

Thus the best strategy for the max player is to take $(y_1, \dots, y_n)$ to be the optimal solution $(y_1^*, \dots, y_n^*)$ of the dual LP and to take

$$\lambda_j = c_j - \sum_i a_{ij} y_i^* \text{ for } j = 1, \dots, k.$$

This makes the value of Game 2 $\sum_i b_i y_i^* = OPT_{dual}$ $\square$

Let us now show a new proof of weak duality theorem.

Claim 3. Value of Game 1 $\geq$ Value of Game 2.

Proof. Suppose the min player chooses values $x_1^\circ, \dots, x_k^\circ$ in Game 1 and the max player chooses values $y_1^\circ, \dots, y_n^\circ, \lambda_1^\circ, \dots, \lambda_k^\circ$ in Game 1.

Suppose the max player chooses values $y'_1, \dots, y'_n$, $\lambda'_1, \dots, \lambda'_k$ in Game 2 and the min player chooses values $x'_1, \dots, x'_k$ in Game 2.

Then $$\underbrace{\sum_j c_j x_j^\circ + \sum_i y_i^\circ \Big(b_i - \sum_j a_{ij} x_j^\circ\Big) - \sum_j \lambda_j x_j^\circ}_{\text{this is the value of Game 1}}$$

$$\geq \sum_j c_j x_j^\circ + \sum_i y'_i \Big(b_i - \sum_j a_{ij} x_j^\circ\Big) - \sum_j \lambda_j x_j^\circ$$

since the max player could as well have chosen $(y'_1, \dots, y'_n, \lambda'_1, \dots, \lambda'_k)$ in Game 1 but he instead chose $(y_1^\circ, \dots, y_n^\circ, \lambda_1^\circ, \dots, \lambda_k^\circ)$. Recall that the max player always desires to maximize the value of a game.

We can rewrite the above sum as

$$\sum_i b_i y'_i + \sum_j x_j^\circ \Big(c_j - \sum_i a_{ij} y'_i - \lambda'_j\Big)$$

$$\geq \sum_i b_i y'_i + \sum_j x'_j \Big(c_j - \sum_i a_{ij} y'_i - \lambda'_j\Big) \quad \text{[this is the value of game 2]}.$$

since in Game 2 the min player could as well have chosen $(x_1^\circ, \dots, x_k^\circ)$ but he instead chose $(x'_1, \dots, x'_k)$. Recall that the min player always desires to minimize the value of a game.

Hence value of Game 1 $\geq$ value of Game 2, i.e., $OPT_{primal} \geq OPT_{dual}$. $\square$

Observe that weak duality or what we proved in Claim 3 is very intuitive – the player who plays second has an advantage in any game since he can see the values chosen by the first player and choose his values accordingly. Since the max player plays second in Game 1 and the min player plays second in Game 2, it is not surprising that value of Game 1 $\geq$ value of Game 2.

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Our goal now is to show that there is an assignment of values to $y_1, \ldots, y_n, \lambda_1, \ldots, \lambda_k$ so that value of Game 2 $= \sum_j c_j x_j^*$ where $(x_1^*, \ldots, x_k^*)$ is the primal optimal solution.

Let $X$ be the total constraint matrix of the primal LP. $X = \begin{bmatrix} A \\ I \end{bmatrix}$ Permute the rows of $X$ into $\begin{bmatrix} X_= \\ X_> \end{bmatrix}$ so that the top $k$ rows are satisfied as equalities by $(x_1^*, \ldots, x_k^*)$

Assume wlog that these rows are the first $t$ rows of $A$ and the remaining $k-t$ rows are some unit vectors. So we have

$$\underbrace{\begin{bmatrix} X_= \end{bmatrix}}_{k \times k \text{ matrix}} \begin{bmatrix} x_1^* \\ \vdots \\ x_k^* \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Call the rows of $X_=$ $\underbrace{\vec{N_1}, \ldots, \vec{N_k}}$ these are the normals of the $k$ hyperplanes whose intersection is $(x_1^*, \ldots, x_k^*)$.

The $k$ vectors $\vec{N_1}, \ldots, \vec{N_k}$ are linearly independent.

So $(C_1, \ldots, C_k)$ can be expressed as a linear combination of $\vec{N_1}, \ldots, \vec{N_k}$. That is, $[C_1 \ldots C_k] = [\beta_1 \ldots \beta_k] \begin{bmatrix} \vec{N_1} \\ \vdots \\ \vec{N_k} \end{bmatrix}$.

Let us take the following assignment of values to $y_1, \ldots, y_n, \lambda_1, \ldots, \lambda_k$:

- each $y_i$ or $\lambda_j$ corresponds to a constraint in the primal LP or equivalently to a row in $X$.

Take $y_i$ or $\lambda_j$ to be $\beta_\ell$ if the corresponding constraint is $\vec{N_\ell}$; otherwise take it to be $0$.

We now need to show the following:
(i) all $y_i$'s and $\lambda_j$'s are $\geq 0$.
(ii) the assignment of values given above makes the value of Game 2 $= \sum_j c_j x_j^*$

Let us show (ii) first. We need to check that our assignment of values to $\lambda_j$'s and $y_i$'s satisfies

$$\sum_i a_{ij} y_i + \lambda_j = c_j \text{ for } j = 1, \ldots, k.$$

Consider the product of 3 matrices

$$[\beta_1 \ldots \beta_k \underbrace{0 \ldots 0}_{\text{n of them}}] \underbrace{\begin{bmatrix} \vec{N_1} \\ \vdots \\ \vec{N_k} \\ X_> \end{bmatrix}}_{\text{this is an } (n+k)\times k \text{ matrix}} \begin{bmatrix} x_1^* \\ \vdots \\ x_k^* \end{bmatrix}.$$

Observe that this product $= \sum_j c_j x_j^*$ by multiplying the first two matrices first and their product with the third.

The product of the first two matrices is $[c_1 \ldots c_k]$.

We claim this product also equals $\sum_{i=1}^{n} b_i y_i$. This is by multiplying the second and third matrices first and then multiplying the first matrix with this product.
The product of the second and third matrices is

$$\begin{bmatrix} b_1 \\ \vdots \\ b_t \\ 0 \\ \vdots \\ 0 \\ * \\ * \end{bmatrix} \begin{array}{l} \} \text{ t coordinates} \\ \} \text{ k-t coordinates} \\ \} \text{ n remaining coordinates} \end{array}$$

The product of the first matrix with this matrix $= \sum_{i=1}^{t} \beta_i b_i = \sum_{i=1}^{n} y_i b_i$ (since the remaining n-t $y_i$'s are 0)

This is t

That is, the value of Game 2 $= \sum_{j=1}^{k} c_j x_j^*$. We will prove (i) in the next lecture.
(assuming non-negativity of $y_i$'s, $\lambda_j$'s)