Weights at the Bottom Matter When the Top is Heavy

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Joint work with Arkadev Chattopadhyay (TIFR)

Outline of talk

Sign Rank

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$rk(M) = 4.$ $\qquad$ $rk(A) = 2.$

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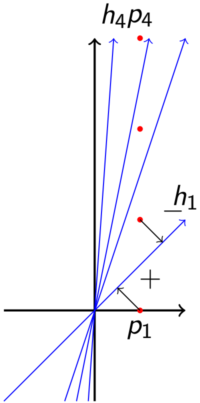

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- Computational learning theory.

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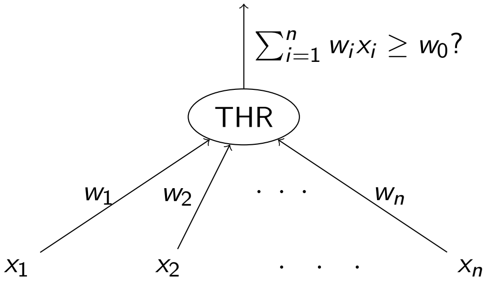

- Can assume $w_i \in \mathbb{Z}$ and $|w_i| \leq 2^{O(n \log n)}$ for all i.
- LT_1 = functions computable by a THR gate.
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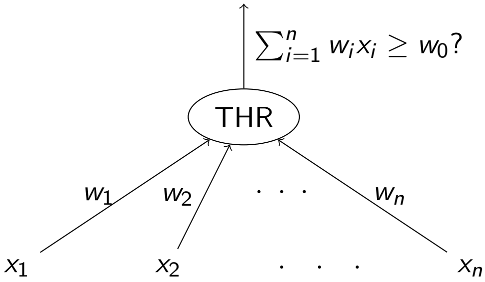

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- There are functions that require weight $2^{\Omega(n \log n)}$ [Håstad '94].
- Limitations of linear thresholds: Parity $\notin LT_1$ [MP '69].

Depth-2 Networks

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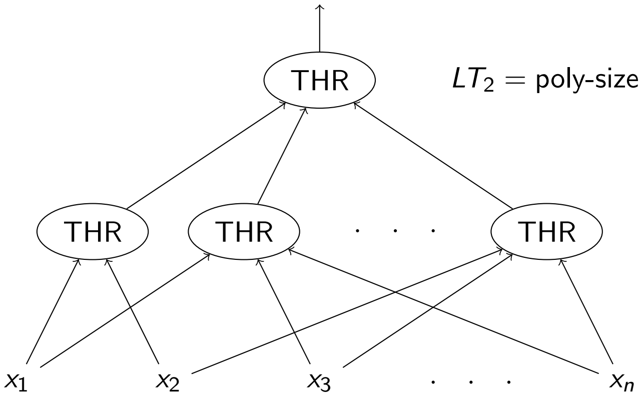

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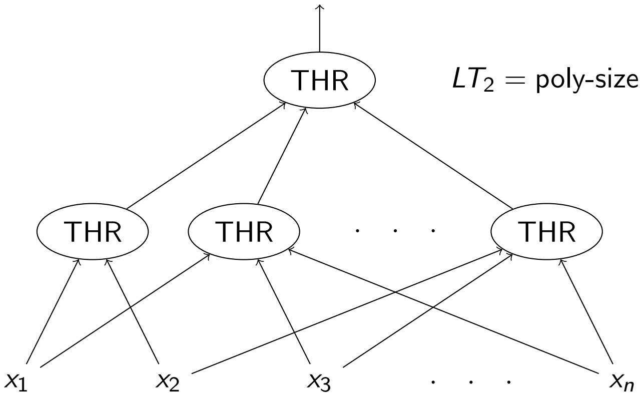

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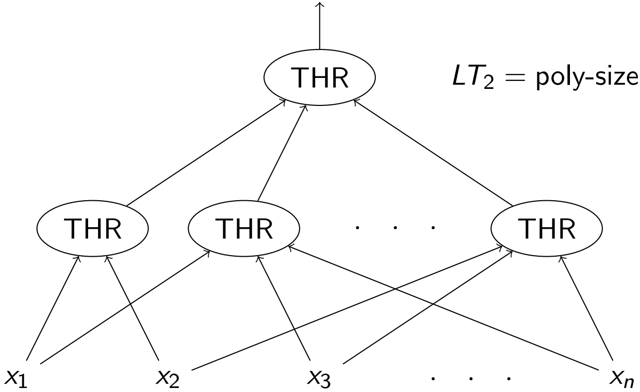

LT_2 = poly-size THR $\circ$ THR

Question: What can LT_2 compute?
Can compute Parity, all symmetric functions.

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Conjecture: $IP_2 \notin LT_2$.

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- Can similarly define the classes $\mathrm{MAJ} \circ \mathrm{THR}$, $\mathrm{THR} \circ \mathrm{MAJ}$.

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Corollary

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Summarizing,

$$\widehat{LT}_1 \subsetneq LT_1 \subsetneq \widehat{LT}_2 = \mathrm{MAJ} \circ \mathrm{THR} \subsetneq \mathrm{THR} \circ \mathrm{MAJ} \subseteq LT_2 \subseteq \widehat{LT}_3.$$

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- Which of the last two containments (possibly both) are strict?
- $\mathrm{THR} \circ \mathrm{MAJ} \subsetneq LT_2$? Open for a long time. Explicitly posed by [AM '05, HP '10].

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Theorem (Chattopadhyay, M '17)

There exists a function $F : \{0,1\}^n \times \{0,1\}^n \to \{-1,1\}, F \in \mathrm{THR} \circ \mathrm{THR}$ *such that* $sr(F) \geq 2^{\Omega(n^{1/4})}$.

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- Recall that $\mathrm{MAJ} \circ \mathrm{MAJ} = \mathrm{MAJ} \circ \mathrm{THR}$ [GHR '92].
- Summary: While weights at the bottom do not matter if the top is light, they do matter if the top is heavy.

XOR functions

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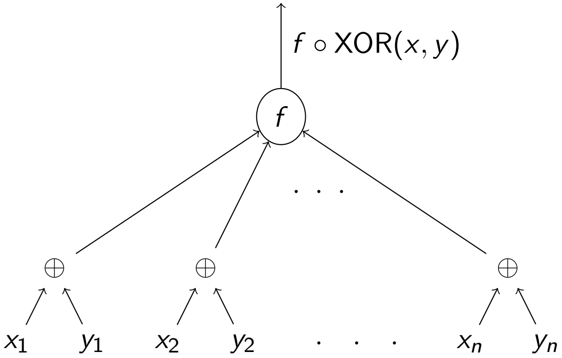

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- Log-rank conjecture open for XOR functions.
- Only two recent works on unbounded error communication complexity of XOR functions [HQ '17, AFK '17], they involve a reduction to pattern matrices [She '11].

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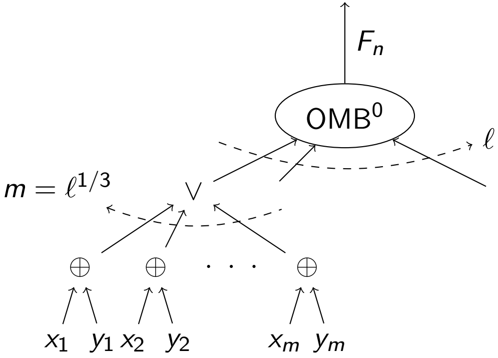

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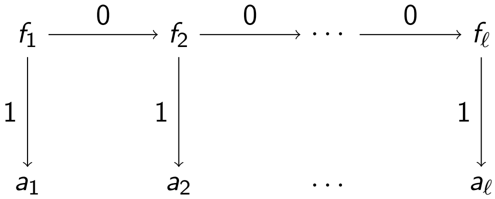

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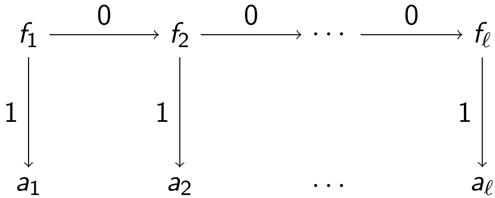

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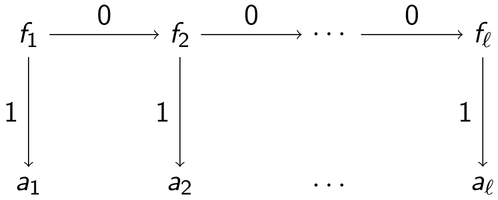

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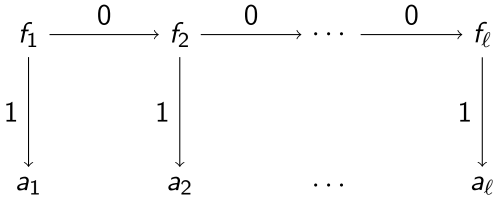

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- $a_i = (-1)^i$.

Model of Communication

$$F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

Alice
$X \in \{0,1\}^n$
R_A

Bob
$Y \in \{0,1\}^n$
R_B

Figure : A protocol Π

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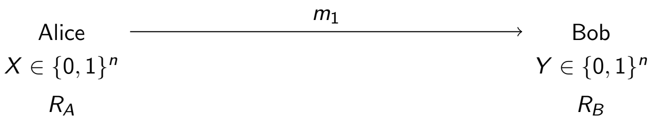

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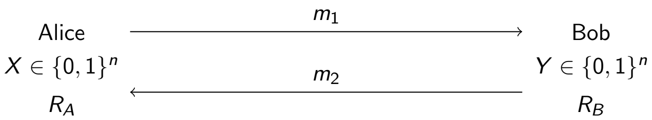

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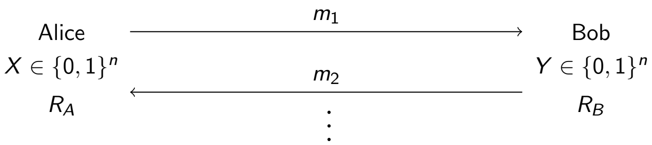

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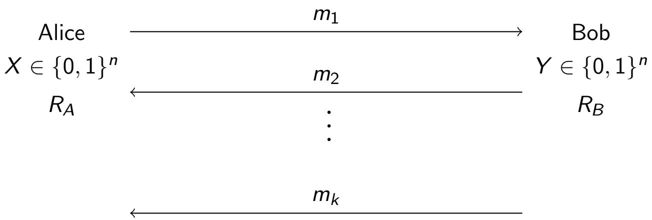

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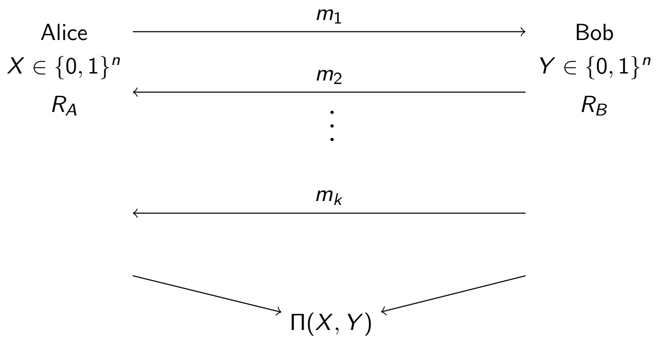

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- $\mathrm{UPP}^{cc} = \{F : \mathrm{UPP}(F) = \mathrm{polylog}(n)\}$.

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- Question: Against what classes does the sign rank method suffice to prove lower bounds?

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First conjecture proved by [BCHTV '17], though for a partial function.

Possible limitations of sign rank

Conjecture (GPW '16)

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- $\mathsf{S_2P} \not\subseteq \mathsf{UPP}$.

First conjecture proved by [BCHTV '17], though for a partial function.

Theorem (Chattopadhyay, M '17)

The total function F_n witnesses

$$\mathsf{P}^{\mathsf{MA}} \not\subseteq \mathsf{UPP}.$$

(It is known that $\mathsf{P}^{\mathsf{MA}} \subseteq \mathsf{S_2P}$).

The class MA

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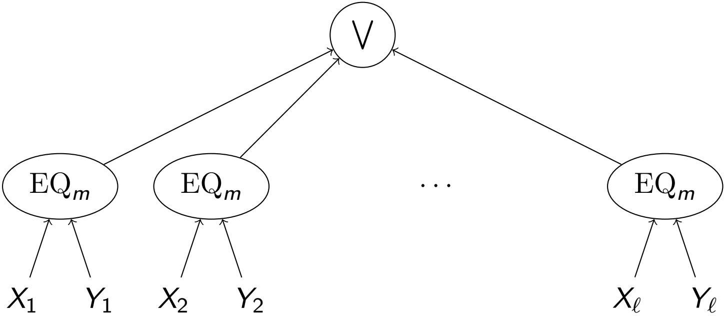
- Correctness: For all inputs, probability of protocol outputting right answer must be at least $2/3$.

An MA protocol for OR ∘ EQ



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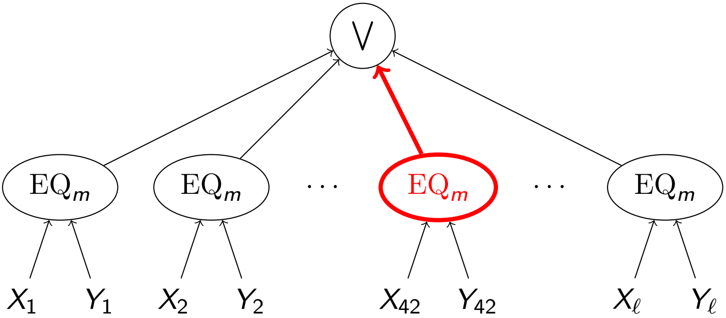

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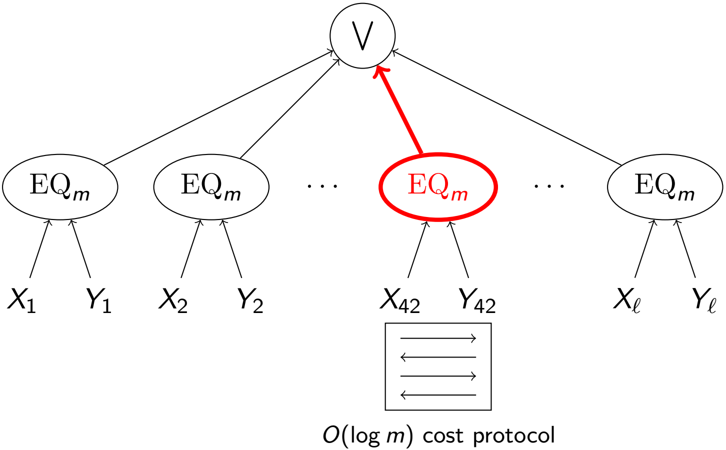

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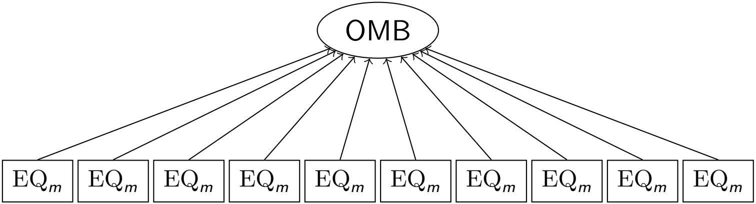
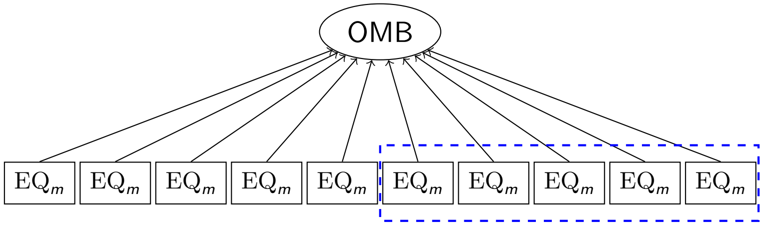
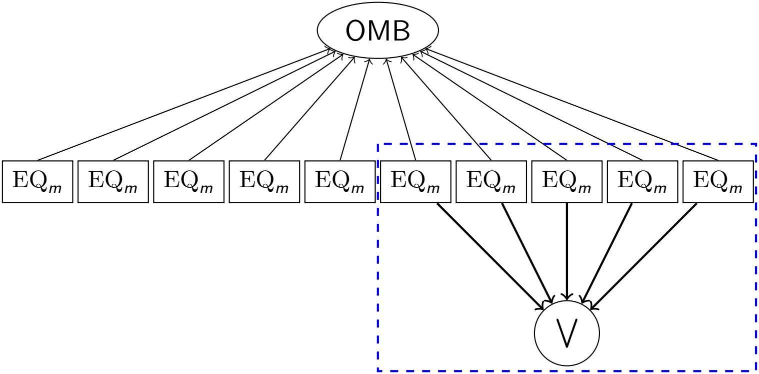
A P^{MA} protocol for OMB ∘ EQ



A P^{MA} protocol for OMB $\circ$ EQ



A P^{MA} protocol for $OMB \circ EQ$



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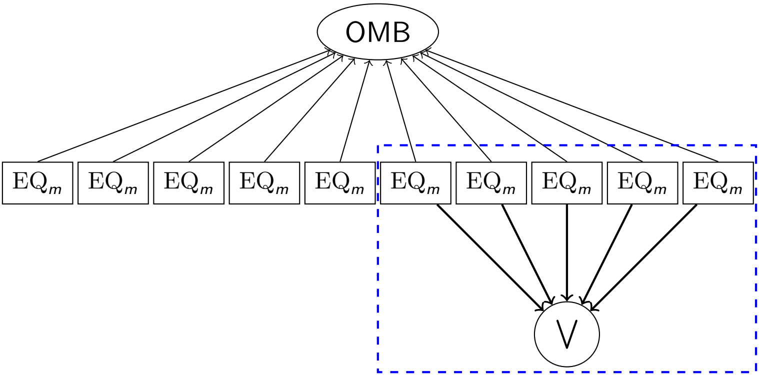
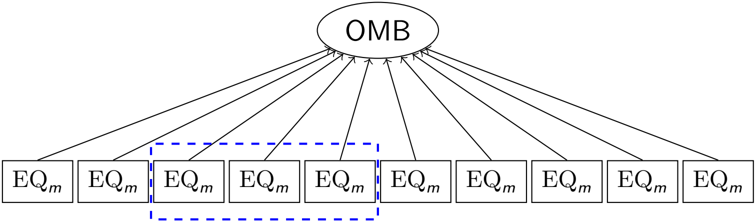


"No"

A P^{MA} protocol for OMB ∘ EQ



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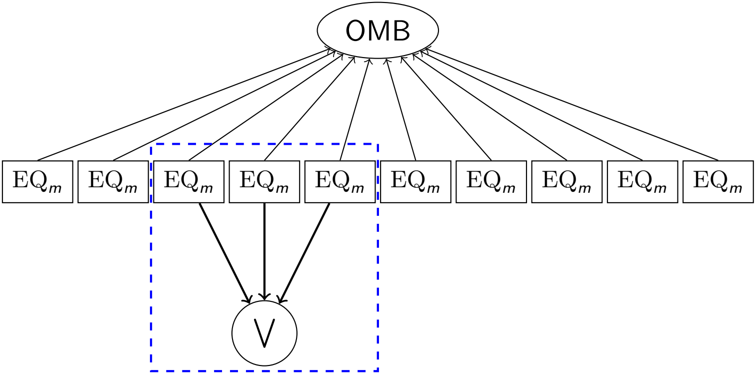

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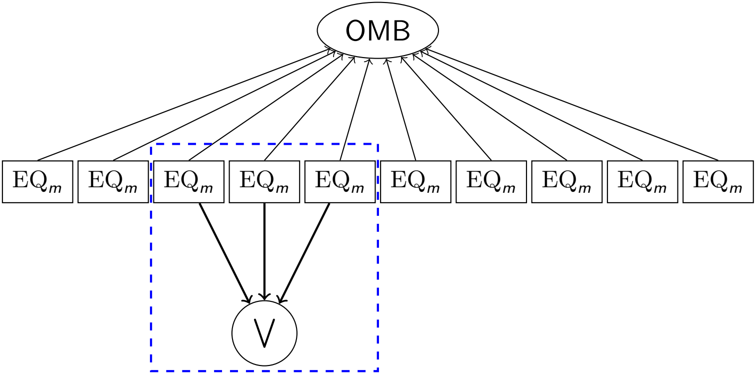

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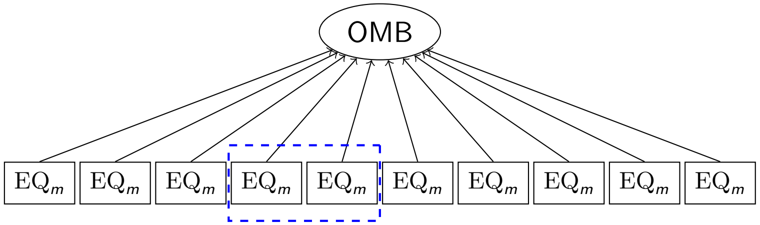

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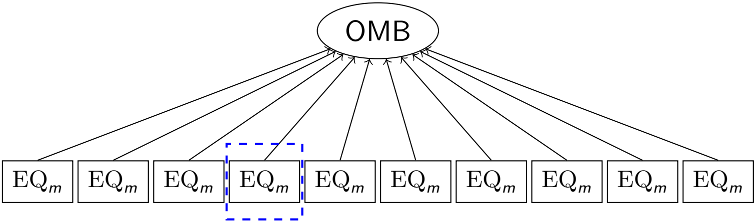

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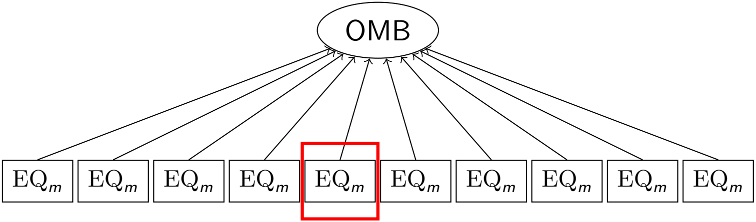

(Mini) Landscape of Communication Classes

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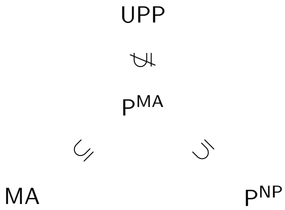

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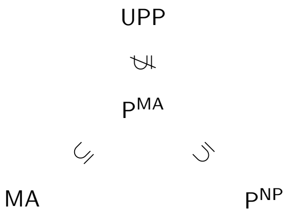

- MA $\subsetneq$ UPP, $P^{NP} \subsetneq$ UPP, lower bound techniques known against MA and P^{NP}.

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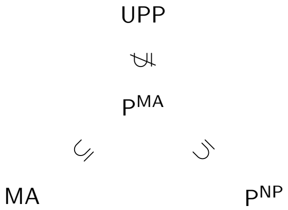

- $\mathsf{MA} \subsetneq \mathsf{UPP}, \mathsf{P}^{\mathsf{NP}} \subsetneq \mathsf{UPP}$, lower bound techniques known against MA and P^{NP}.
- Natural program: Find lower bound techniques against P^{MA}.

Reducing to an approximation theoretic problem

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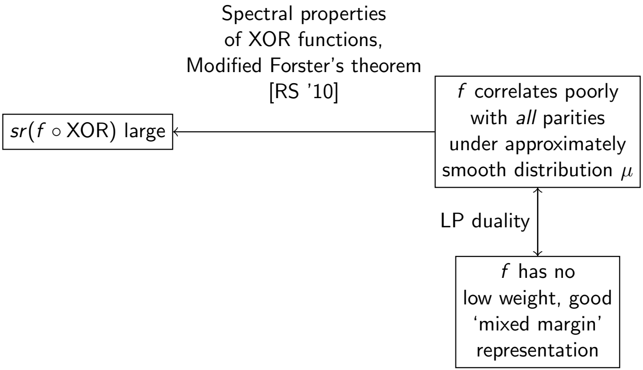

Figure : Approximation theoretic hardness of f implies large sign rank of $f \circ \text{XOR}$.

An LP Formulation

An LP Formulation

Variables	$\epsilon, \{\mu_x : x \in \{-1,1\}^n\}$		
Minimize	ϵ		
s.t.	$\left\| \sum_x \mu(x) f(x) \chi_S(x) \right\|$	$\leq \epsilon$	$\forall S \subseteq [n]$
	$\sum_x \mu(x)$	$= 1$	
	$\epsilon \geq 0$		
	$\mu(x) \geq \frac{\delta}{2^n}$		$\forall x \in X$
	$\mu(x) \geq 0$		$\forall x \in \{-1,1\}^n$

An LP Formulation

$$
\begin{array}{lll}
\text{Variables} & \epsilon, \{\mu_x : x \in \{-1,1\}^n\} & \\
\text{Minimize} & \epsilon & \\
\text{s.t.} & \left|\sum_x \mu(x) f(x) \chi_S(x)\right| \leq \epsilon & \forall S \subseteq [n] \\
 & \sum_x \mu(x) = 1 & \\
 & \epsilon \geq 0 & \\
 & \mu(x) \geq \frac{\delta}{2^n} & \forall x \in X \\
 & \mu(x) \geq 0 & \forall x \in \{-1,1\}^n
\end{array}
$$

Sign rank lower bound obtained:

$$
sr(f \circ \text{XOR}) \geq \frac{\delta}{OPT + \delta \cdot \frac{|X^c|}{2^n}}.
$$

LP Dual

LP Dual

Variables $\Delta, \{\alpha_S : S \subseteq [n]\}, \{\xi_x : x \in X\}$

Maximize $\Delta + \frac{\delta}{2^n} \sum_{x \in X} \xi_x$

s.t.

$$
\begin{array}{lll}
f(x) \sum_{S \subseteq [n]} \alpha_S \chi_S(x) & \geq \Delta & \forall x \in \{-1,1\}^n \\
f(x) \sum_{S \subseteq [n]} \alpha_S \chi_S(x) & \geq \Delta + \xi_x & \forall x \in X \\
\sum_{S \subseteq [n]} |\alpha_S| & \leq 1 & \\
\Delta \in \mathbb{R} & & \\
\alpha_S \in \mathbb{R} & & \forall S \subseteq [n] \\
\xi_x \geq 0 & & \forall x \in X
\end{array}
$$

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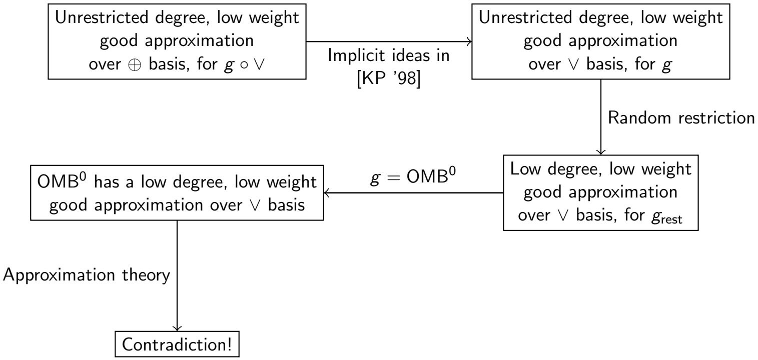

Figure : Approximation theoretic analysis

Thank You!