Analysis of An Adaptive Sampler Based on Weber’s Law

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Abstract

Weber’s law suggests a logarithmic relationship between perceptual stimuli and human perception. The Weber sampler is an adaptive, non-uniform sampling mechanism that exploits Weber’s law to sample the signal at a minimum rate without significant perceptual degradation. In this paper, we introduce and analyze a regularized Weber sampler for smooth deterministic signals as well as smooth random processes. While analysis for a fixed Weber constant $\delta$ is analytically intractable except for some special cases, we exploit the fact that the Weber constant is usually small. Under suitable assumptions, for deterministic signals, we provide analytical approximations to the number of samples in a unit time interval and the inter-sample times. For a random signal, we give analytical approximations to the expectation and probability distribution function of the respective quantities. Our class of deterministic as well as random signals is quite wide, and in particular covers bandlimited signals and signals driven by second order ordinary differential equations. We also present a number of simulations to demonstrate that our approximations are good up to a Weber constant of 0.2, which is the regime of practical interest. Our results provide a source model for the Weber sampler, which can be used in the study of transmission of perceptual stimuli signals over communication links.

I. INTRODUCTION

Motivated by applications such as interactive games, telepresence, etc., there has recently been vibrant activity in studying the transmission of haptic signals over existing communication networks such as the internet. Such transmission is best carried out by sampling the analog haptic signal and digitizing it. Given the strict delay constraints for applications such as interactive games, it is of interest to design sampling mechanisms that result in low transmission rate but at the same time yield a good perceptual experience at the remote location. While traditional designs employ periodic sampling, recently some authors have tried to exploit Weber’s law ([1], [2]), which states that human perception depends on percentage changes with respect to current signal levels. A sampling mechanism based on Weber’s law is not periodic and it emits a sample when the percentage change in the signal level compared to the previous sample crosses a pre-determined threshold. We refer to such a sampling mechanism as the Weber sampler. Such a sampler is not only of interest for haptic signals, but might also be useful in visual and auditory perception, where again Weber’s law is known to hold ([4], [5], [6]). Even though the Weber sampler arises naturally from the Weber law, its analysis turns out to be difficult, and few analytical results are known ([1], [3]). In this paper, we partly fill this gap by analyzing
the effective sampling rate and inter-sample time of the Weber sampler for a broad class of
deterministic as well as random signals. Such an analysis provides a “packet generation model”
for a signal source sampled with the Weber sampler. It has the potential to be used along with
aspects of the communication network to determine quality of perception, resource provisioning
in the network, etc. In the remainder of this section, we describe prior related work and we
outline our contribution in the context of these works.

A. Prior Related Work

The Weber-Fechner law - a corollary of the Weber’s law - essentially states that the relationship
between a stimulus and its perception follows a logarithmic relationship. Accordingly if the signal
level is $s$, then only those changes for which the new signal level $x$ satisfies $|x - s|/|s| > \delta$ are
perceived, where $\delta$ is called Weber’s constant or the just noticeable difference (JND). There is
a large body of work relating Weber’s law and different forms of perception. For example, for
a visual stimulus, it has been shown by Stiles [5] that an isolated human cone photoreceptor
has an incremental threshold corresponding to the Weber constant 0.018 for photopic vision.
However, this value goes up to 0.06 when the perceived brightness comes from a grating patch
[4]. Thus if there is a local variation in intensity pattern smaller than 6%, then it cannot be
perceived by humans. For the purpose of auditory perception, according to [6], Weber’s law
emerges by considering the power of a sound source (analogous to the image reflectance). For
the detection of change in a wideband auditory signal, it is shown in [7] that the corresponding
Weber constant varies between 0.12 to 0.24. In [8], statistical signal analysis is used to compute
the limits for the detection of noticeable change in auditory signal. Weber’s law has also been
explored in the context of odour [9], [10] and temperature [11]. Touch, unlike other types of
sensing, is an active sensing mechanism: pressing an object elicits a reactive force, which causes
the haptic sensation. It has also been shown to follow Weber’s law [12], [13] and the Weber
constant is claimed to be 0.15 [14]. However, it should be noted that there are cross-linkages of
different types of stimuli in the overall perception. For example, see [15], which discusses how
both the visual and the haptic perceptions play important roles for detection and discrimination
of objects.

Given the wide validity of Weber’s law, it is not surprising that several researchers have
attempted to exploit it for quantizing and sampling analog signals. The idea of JND has been
used in speech parameter quantization long ago [16], [17]. In [18], the JND has been used to analyze the performance of transform domain quantizers. A detailed survey of the application of JND for audio, video signal compression can be found in [19]. Similar ideas have also been used for quantization of the color space in [20]. Thus for the application of Weber’s law in the context of quantization, there is a wide body of literature, which typically uses periodic sampling of the signals. While there is a wide body of literature on non-uniform sampling of signals (see for example [23], [24], [25], [26]), the exploitation of Weber’s law for adapting the sampling time instants has received relatively less attention. In graphics, the logarithmic relationship that defines perception has been used effectively to reduce the number of meshes [21], and is conceptually related to the Weber sampler we analyze. In image analysis, a wavelet based adaptive sampling image grid is proposed in [22] so that perception of contrast is not compromised. For transmission of haptic signals over a communication network, Weber’s law has been used for adaptive sampling in [1], [2], and it is reported to lead to 80-90% reduction in the packet rate. Thus the Weber sampler can be very effective. But few efforts have been made to analyze its performance. One such effort is [3], where the savings in packet rate for the Weber sampler are predicted for signals that live on a fine discrete grid and have independent and identically distributed samples. However, to fully understand the Weber sampler, we need to study its effect on continuous time signals from a wide class of signals. In this paper, we address this void. In the following section, we briefly describe our contribution.

B. Our Contribution and Organization of Paper

In this paper, we analyze the Weber sampler for one-dimensional, real-valued deterministic as well as random continuous time signals. For a fixed but arbitrary Weber constant $\delta$, the analysis usually does not lead to any clean results. However, in practice, we see that the Weber constant $\delta$ is usually small. We exploit this fact and give analytical approximations for various quantities, which are found to be accurate up to $\delta = 0.2$, which is the regime of practical interest.

In Section II, we define the Weber sampler for a signal $f(t)$, $t \geq 0$, and give a simple example to show that the Weber sampler can lead to an infinite sampling rate. Motivated by this, we propose a regularized Weber sampler, which we analyze in the rest of the paper. In Section III, we analyze the Weber sampler for deterministic signals, and we obtain analytical approximations for the number of samples in an unit interval ($N_\delta$) and the inter-sample time.
(that is, the time between two successive samples). Our main result for smooth deterministic signals is given in Section III-A, which are extended to signals with discontinuities in III-B. We also give a number of examples, including simulation results, to demonstrate the accuracy of our approximations for common signal models such as bandlimited signals and signals satisfying second order ordinary differential equations. The reconstruction error of the Weber sampler under linear interpolation is given in Section III-C.

In Section IV, we analyze the Weber sampler for smooth random signals. Our focus is on identifying the intensity of the sampling process $\lambda_\delta$ (that is, the mean of the number of samples per unit time) and the probability law of the inter-sample times. While renewal models such as the Poisson process are commonly sought after, we show that even for a Markov process, the inter-sample times of the Weber sampler are dependent. However, conditioned on the samples, we show independence, and we derive approximations to the conditional marginal distribution function. We also present analytical approximations to $\lambda_\delta$. We also compare our analytical results with estimates obtained from Monte-Carlo simulation for stationary bandlimited processes, and demonstrate the accuracy of our approximations.

The conclusion is given in Section V and the proofs are given in Section VI.

II. THE WEBER SAMPLER AND ITS REGULARIZATION

Consider a real-valued signal $f(t)$, $t \in [0, 1]$. The Weber sampler outputs samples at times $T_n$, $n \geq 0$ given by the following rule:

$$
T_0 := 0
$$

$$
T_n := \inf \left\{ t \in (T_{n-1}, 1] : \frac{|f(t) - f(T_{n-1})|}{|f(T_{n-1})|} > \delta \right\}, \quad n \geq 1,
$$

where $\delta > 0$ is the Weber constant or the just noticeable difference (JND), which influences human perception of the signal. In words, if the previous sample is at time $T_{n-1}$, then the next sample is taken at the earliest future time instant for which the relative change $|f(t) - f(T_{n-1})|/|f(T_{n-1})|$ exceeds the JND $\delta$. If we use a threshold smaller than the Weber constant $\delta$, then as per the Weber law, there is no noticeable change in perception. However, if we choose a threshold larger than the Weber constant, then the perception degrades. Thus the Weber sampler is an adaptive, non-uniform sampling scheme that seeks to sample at the least rate that results in acceptable human perception of the signal.
In this paper, our aim is to analyze the following three quantities.

- The total number of samples generated by the Weber sampler in a given interval $[0, 1]$, which is defined as
  \[ N_\delta = \max\{n \geq 0 : T_n \leq 1\}. \]

- The sequence of inter-sample times: $\{T_n - T_{n-1}, n \geq 1\}$.

- The reconstruction error
  \[ e(\delta; t) = \frac{|\hat{f}(t) - f(t)|}{\max\{|\delta_0, \delta_1|f(t)|\}}, \]
  where $\hat{f}(t)$ denotes the estimate of $f(t)$ obtained by linear interpolation of the samples and the parameters $\delta_0$, $\delta_1$ are explained on the next page.

In general, these quantities depend on $f(t)$ in a complex manner, and except for some special cases, explicit computation is not feasible. But in practice, the Weber constant is usually small (typically $\delta < 0.15$), and there is hope to obtain a simple approximation for small $\delta$. We pursue this angle in this paper.

But before we can proceed with the analysis, we note that we need to modify the original Weber sampler. The Weber sampler described above is an idealized sampler motivated by experiments ([1], [2]). In this idealized form, it can result in an infinite number of samples in a finite interval even for very simple cases. For example, consider the following.

**Example :** Consider the function $f(t) = 1 - t$. Since the function is continuous, decreasing and non-negative, we get that

\[ f(T_{n-1}) - f(T_n) = \delta f(T_{n-1}) \]

that is

\[ T_n - T_{n-1} = \delta(1 - T_{n-1}). \]

Thus we get the recursion $T_n = (1 - \delta)T_{n-1} + \delta$. Using the initial condition $T_0 = 0$, we get

\[ T_n = 1 - (1 - \delta)^n. \]

We see that for any $\delta \in (0, 1)$, $T_n < 1$ for any finite $n$ and thus $N_\delta = \infty$. 

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Fortunately, this example is avoided in any practical implementation of the Weber sampler since in a finite precision implementation it is not possible to identify arbitrarily small differences. Motivated by this, we consider the following regularized Weber sampler:

\[
T_0 := 0 \\
T_n := \inf\left\{ t \in (T_{n-1}, 1] : |f(t) - f(T_{n-1})| > \max\{\delta_0, \delta_1|f(T_{n-1})|\}\right\}, \quad n \geq 1.
\]

This sampler has two parameters \(\delta_0, \delta_1\). If we put \(\delta_1 = 0\), then we get a level-crossing sampler, while if we put \(\delta_0 = 0\), then we get the Weber sampler above. In the rest of this paper, we use the term “Weber sampler” to refer to the above regularized version. We choose \(\delta_1 = \delta\) (the Weber constant) and \(\delta_0 = \alpha \delta\), where \(\alpha > 0\) is a fixed constant.\(^1\) Our interest is in the case when \(\alpha << 1\). The choice of \(\alpha\) is dictated by two factors:

1) In a finite precision implementation, it is not possible to distinguish between small values and \(\delta_0\) needs to be chosen larger than the smallest number representable in the given precision.

2) In the presence of noise, \(\delta_0\) ensures that the sampler does not react to small wiggles in the signal due to noise. For example, if the noise is zero mean Gaussian with variance \(\sigma^2\), then noise deviations are in \([-3\sigma, 3\sigma]\) with a high probability, and we should choose \(\delta_0 \geq 3\sigma\).

**Remark:** While the above example showed that the Weber sampler can lead to an infinite sampling rate for simple cases, the other extreme of zero rate occurs for constant signals. For general signals with points of zero derivative, this means that the sampling rate can potentially be arbitrarily small. When the signal has to be communicated to a distant location, in order to keep the communication link alive, it is common to maintain a minimum data rate over the link using dummy packets if needed. In such a case, it is better to maintain a minimum sampling rate and transmit useful information instead of dummy packets. Thus, samples may be taken at times

\[
T_n := \min \left\{ \inf\left\{ t \in (T_{n-1}, 1] : |f(t) - f(T_{n-1})| > \max\{\delta_0, \delta_1|f(T_{n-1})|\}\right\}, T_{n-1} + \frac{1}{R_{\text{min}}} \right\}
\]

\(^1\)We note that \(\delta\) is a unitless quantity while \(\alpha\) has the same unit as \(|f(t)|\).
where $R_{\text{min}} > 0$ is the minimum rate desired. The analysis of this variant of the Weber sampler follow directly from our analysis of the Weber sampler. We do not pursue this variant any more in this paper.

III. Analysis for Deterministic Signals

In this section, we consider a deterministic signal $f(t)$ defined on an interval $[0, 1]$. Deterministic signals are not only of interest in their own right, but their analysis also forms the backbone of our analysis for random signals in Section IV. In Section III-A, we state our main result for smooth signals, which is extended to signals with discontinuities in Section III-B. The reconstruction error for linear interpolation is given in Section III-C.

A. Smooth Signals

Under some smoothness assumptions on $f(t)$, we show that for small $\delta_0, \delta_1$,

$$N_\delta \approx \left[ \int_0^1 \frac{|f'(t)|}{\max\{|\delta_0, \delta_1|f(t)|\}} \, dt \right]$$

where the square brackets denote rounding to the closest integer, and,

$$T_n - T_{n-1} \approx \frac{\max\{|\delta_0, \delta_1|f(T_{n-1})|\}}{|f'(T_{n-1})|}.$$

The exact meaning of the approximation $\approx$ is made clear in Theorem 1 below, which is followed by a detailed discussion including some numerical results demonstrating the accuracy of the approximations. But before stating the result, we state and discuss our assumptions. We make the following assumptions to establish the above approximations.

(A1) $f(t) \in C^2([0, 1])$ (that is, it is differentiable two times and the second derivative is continuous);

(A2) At any point $t \in [0, 1]$, both the first and second derivatives cannot be zero.

While (A1) demands the signal to be two times differentiable, we show later in Theorem 2 that we can accommodate a finite number of critical points (points of discontinuity of signal and its derivatives). Assumption (A2) rules out functions of the type $t^3$ (which has zero derivatives up to order two at $t = 0$). We can relax (A2) to the assumption that at each point the signal has a non-zero derivative of some order. However such an assumption complicates the proof considerably without providing any additional insight. Hence we prefer to work with (A2) instead. Under
these assumptions, our main result is the following.

**Theorem 1:** Consider a deterministic signal \( f(t), t \in [0, 1] \) satisfying assumptions (A1) and (A2) above. Let \( \delta_0 = \alpha \delta, \delta_1 = \delta, \alpha > 0 \), and consider the limit \( \delta \to 0 \) (with \( \alpha \) fixed).

1) Let \( N_\delta \) denote the number of output samples of the Weber sampler in the interval \([0, 1]\).

   Then
   \[
   \lim_{\delta \to 0} \delta N_\delta = \int_0^1 \frac{|f'(t)|}{\max\{\alpha, |f(t)|\}} dt. \tag{1}
   \]

2) For \( t \in [0, 1] \) define
   \[
   T(\delta; t) := \inf \left\{ u \in (t, 1] : \frac{|f(u) - f(t)|}{\max\{\alpha, |f(t)|\}} > \delta \right\}.
   \]

   If \( f'(t) \neq 0 \), then
   \[
   \lim_{\delta \to 0} \frac{T(\delta; t)}{\delta} = \frac{\max\{\alpha, |f(t)|\}}{|f'(t)|}
   \]

   and otherwise
   \[
   \lim_{\delta \to 0} \frac{T(\delta; t)}{\sqrt{\delta}} = \sqrt{\frac{2 \max\{\alpha, |f(t)|\}}{|f''(t)|}}.
   \]

**Proof:** The proof is given in Section VI-A.

**Discussion of Part 1 of Theorem 1:** Part 1) of Theorem 1 says that for the modified Weber sampler, for small \( \delta_0, \delta_1 \), the number of output samples is approximately

\[
N_{\text{approx}} := \left[ \int_0^1 \frac{|f'(t)|}{\max\{\delta_0, \delta_1 |f(t)|\}} dt \right]
\]

where the square brackets denote rounding to the closest integer. For some simple cases, \( N_{\text{approx}} \) can be evaluated in closed form, but in general we may need numerical methods for integration. For example, for \( f(t) = 1 - t \),

\[
N_{\text{approx}} = \left[ \frac{1}{\delta_1} \left( 1 - \ln \left( \frac{\delta_0}{\delta_1} \right) \right) \right].
\]

In fact, the exact \( N_\delta \) for \( f(t) = 1 - t \) is given by

\[
N_\delta = \left[ \frac{\ln(\delta_0/\delta_1)}{\ln(1-\delta_1)} + \frac{1}{\delta_1} \right].
\]

We see that if we use the approximation \( \ln(1-\delta_1) \approx -\delta_1 \), then we get the expression for \( N_{\text{approx}} \) above.
In practice, even though $\delta_0$ and $\delta_1$ are small, they are non-zero. To check the accuracy of the above approximation, next we consider several simulation examples. Consider the solutions of differential equations such as

$$af''(t) + bf'(t) + cf(t) + d(t) = 0,$$

(2)

where $a, b, c$ are parameters governing the response and $d(t)$ is the driving signal (such as the pressure applied). The motivation for using (2) for experimentation purposes is as follows. For a haptic phantom working in the linear zone of the Hooke’s law, the response of the phantom is given by a second order linear differential equation with constant coefficients. The coefficients are usually such that there is enough damping to ensure stability of the system. The component $d(t)$ is the forcing input provided by the user or the reactive force provided by the surface being touched. Hence $d(t)$ depends on the user experience and can vary for different applications.

In Table I, we consider a number of functions that are solutions of such differential equations for various values of $a, b, c, d(t)$ and suitable initial conditions. We compare $N_\delta$ obtained from simulations with the above approximation for different functions and different values of $\delta_1$. We fix $\delta_0 = 10^{-4}$ and for evaluating $N_{\text{approx}}$ we use numerical integration. We see that the match is good, and particularly for small $\delta_1$, the percentage error is small.

Another important class of functions is the class of bandlimited signals. A bandlimited function is infinitely differentiable and hence satisfies (A1), (A2) for any finite interval. As a representative of bandlimited functions, we show the numerical results for $\sin(2\pi t)/(2\pi t)$ in Table I. We again
see that our approximation is quite good.

The above analytical approximation can be used to find the gains expected from a Weber sampler compared to an uniformly spaced sampler. In order to give the same perceptual experience as the Weber sampler, the uniformly spaced sampler has to be designed for the worst case relative change, and hence the number of samples is approximately

\[ N_{\text{uniform}} \approx \sup_{t \in [0,1]} \frac{|f'(t)|}{\max\{\delta_0, \delta_1|f(t)|\}}. \]

Thus, for small \( \delta \), the Weber sampler needs a fraction

\[ G = \frac{\int_0^1 \frac{|f'(t)|}{\max\{\alpha,|f(t)|\}} dt}{\sup_{t \in [0,1]} \frac{|f'(t)|}{\max\{\alpha,|f(t)|\}}} \]

due to the logarithmic relationship with perception, the Weber sampler with \( \alpha = 0 \) reduces to a uniform sampler, and hence \( G = 1 \).

Discussion of Part 2 of Theorem 1: Part 2) of Theorem 1 gives an approximation for the inter-sample time:

\[ T_n - T_{n-1} \approx \frac{\max\{|\delta_0, \delta_1|f(T_{n-1})|\}}{|f'(T_{n-1})|}, \quad \text{if } f'(T_{n-1}) \neq 0. \]

In fact, for the example \( f(t) = 1-t \), we see that the exact expression \( T_n - T_{n-1} = \max\{|\delta_0, \delta_1(1-T_{n-1})|\} \) matches with the above approximation. This is a reflection of the fact that our proof of this result relies on a local linear approximation of the function, and the approximation error is zero for linear functions. To give another example, for \( f(t) = e^{-t} \), the exact value of \( T(\delta; t) \) is

\[ T(\delta; t) = t - \ln \left( 1 - \max\{\delta_0e^t, \delta_1\} \right), \]

while the approximation from Theorem 1 is

\[ T(\delta; t) \approx t + \max\{\delta_0e^t, \delta_1\}, \]

which corresponds to the approximation \( \ln(1 + x) \approx x \) for small \( x \). We note that the behavior of \( T(\delta; t) - t \) depends on whether \( f'(t) = 0 \) or not. For the example \( f(t) = \sin(\pi t)/(\pi t) \), we see that the first derivative is zero at \( t = 0 \), and hence, we get the approximation

\[ T(\delta; 0) \approx \frac{\sqrt{6\delta}}{\pi}. \]
To illustrate the accuracy of this approximation, in Figure 1, we plot this approximation and the \( T(\delta;0) \) obtained by simulating the Weber sampler. Our focus is on \( \delta \) up to 0.2, which is the case for most sensory perception, and we see that our approximation is good up to this value.

**B. Signals with Discontinuities**

Our assumptions (A1) and (A2) are partly inspired by the properties of the solutions of (2). In case of sudden contact with a rigid object or sudden variation in surface characteristics, we expect rapid changes in the haptic signal and its derivatives. In such situations, functions having discontinuities and discontinuous derivatives with smooth behavior in between such critical points may provide a more appropriate model. Theorem 1 can be extended to the case where the number of critical points is finite and assumptions (A1)-(A2) are satisfied in the intervals between the critical points. To understand this, we note that Theorem 1 is applicable to any interval whose size remains fixed as \( \delta \to 0 \). If we could apply it to each of intervals between the critical points,
then by additivity of limits and integrals, the desired extension would follow. We note that the integral on the right hand side of (1) is well-defined for such non-smooth signals, since the integrand is defined at all but a finite number of points. This result is stated formally below and is proved in Section VI-B.

**Theorem 2:** Consider a deterministic signal \( f(t), t \in [0, 1] \) with such that
- it has at most a finite number of points of discontinuity;
- its derivative is defined almost everywhere and has at most a finite number of points of discontinuity;
- assumptions (A1)-(A2) are satisfied in the interval between any two critical points.

Then the conclusions of Theorem 1 are true.

**Proof:** The proof is given in Section VI-B.

### C. Reconstruction Error Analysis

We next study the reconstruction error. We note that in practice, not only do we sample at times \( \{T_n\} \), but the signal level \( f(T_n) \) is also quantized. In the analysis below, we ignore such quantization error, since in typical applications high precision quantization is used and the quantization loss is small. We estimate \( f(t) \) by linear interpolation of the samples \( \{f(T_n), n \geq 0\} \):

\[
\hat{f}(t) = \frac{(t - T_{n-1})}{T_n - T_{n-1}} f(T_n) + \frac{(T_n - t)}{T_n - T_{n-1}} f(T_{n-1}).
\]

In the spirit of Weber’s law, and the regularization discussed in Section II, a natural measure of the reconstruction error at time \( t \) is

\[
e(\delta; t) = \frac{|\hat{f}(t) - f(t)|}{\max\{\delta_0, \delta_1 |f(t)|\}}.
\]

Below we establish a bound on this error.

**Theorem 3:** Suppose assumptions (A1), (A2) are true and \( |f''(t)| \leq L < \infty \). Then for sufficiently small \( \delta \),

\[
\sup_{t \in (T_{n-1}, T_n]} |\hat{f}(t) - f(t)| \leq \frac{L}{2} (T_n - T_{n-1})^2
\]

and consequently for \( t \in (T_{n-1}, T_n] \),

\[
e(\delta; t) \leq \frac{L(T_n - T_{n-1})^2}{2 \max\{\delta_0, \delta_1 |f(t)|\}}.
\]
If in addition, we have that \(|f'(t)| \geq \epsilon > 0\) for \(t \in [0, 1]\), then we get

\[
\sup_{t \in [0,1]} e(\delta; t) \leq C \delta
\]

for a finite positive constant \(C\).

**Proof:** The proof is given in Section VI-C.

Theorem 3 says that the absolute reconstruction error \(|\hat{f}(t) - f(t)|\) in the interval \((T_{n-1}, T_n]\) is proportional to the square of the length of the interval, which is similar to the \(O(\Delta^2)\) obtained by linear interpolation of uniformly spaced samples with spacing \(\Delta\). In the neighborhood of points where \(f'(t) \neq 0\), we expect the interval length to be proportional to \(\delta\) (see Part 2 of Theorem 1). Hence we expect \(e(\delta; t)\) to be proportional to \(\delta\).

**IV. ANALYSIS FOR RANDOM SIGNALS**

Consider an interactive game involving haptic communication. The haptic signal generated at each player depends on the conditions of the game and varies across different instances of the game. Thus our interest is not to design the system for a single haptic signal, but rather for a collection or an ensemble of signals. One common way to deal with ensembles of signals is to consider a random process, which is obtained by putting a probability measure on the ensemble of signals. In this section, we analyze the Weber sampler for random processes.

Consider the transmission of a haptic signal over a communication network. Since low delay is a primary constraint, each sample output by the Weber sampler is sent in a separate packet without waiting for future samples. From the perspective of the communication network, this is an information source generating packets at random times \(\{T_n\}\), and in the communication network literature, typically renewal process/Poisson process models are used to describe such sources [27]. For a stochastic process with smooth sample paths passed through a Weber sampler, below we study the intensity of the sampling points and the probability law governing the inter-sample times.

**Theorem 4:** Consider a random process \(\{X(t), t \geq 0\}\) passed through the Weber sampler. The following are true.

1) Suppose the samples paths of \(\{X(t)\}\) satisfy (A1)-(A2) a.s.. Then \(\delta N_\delta\) converges a.s. (and hence in distribution) to

\[
\int_0^1 \frac{|X'(t)|}{\max\{\alpha, |X(t)|\}} dt.
\]
2) If \{X(t), t \geq 0\} is a Markov process satisfying the strong Markov property [28], then the discrete time process \(X = \{X(T_n), n = 0, 1, 2, \ldots\}\) is a Markov chain, and conditioned on \(X, \{T_n - T_{n-1}, n \geq 1\}\) are independent.

3) Suppose the sample path of \(\{X(t)\}\) satisfies (A1)-(A2) a.s. and suppose that \(X'(t) \neq 0\) a.s. Then \((T(\delta; t) - t)/\delta\) converges a.s. (and hence in distribution) to
\[
\frac{\max\{\alpha, |X(t)|\}}{|X'(t)|}.
\]

**Proof:** The proofs of Part 1) and 3) follow directly from Theorem 1. Part 2) follows from the fact that \(\{T_n\}\) are stopping times, and an application of the strong Markov property. (For readers not familiar with stopping times and the strong Markov property, here we add that a stopping time is a random time instant whose occurrence does not depend on the future, and the strong Markov property states that the conditional independence of the future and past holds even if we condition on the state at a stopping time. For technical details, we refer to [28].)

Parts 1) and 3) require the random process to have smooth sample paths. Conditions that ensure such sample path smoothness can be found in [29, Chapter 4]. Since these are of a technical nature, we do discuss them here, but it is important to mention that this is a wide class. In particular, we can obtain such smooth processes by driving the ODE (2) with a white noise process (that is we replace the RHS of (2) by a white noise process to obtain a stochastic differential equation), which also turns out to have the strong Markov property, required by Part 2). Also we can consider bandlimited stationary processes, which have infinitely differentiable paths that satisfy (A1)-(A2).

We note that \(\lambda_\delta := E[N_\delta]\) is the expected number of samples in unit time, that is, it is the intensity of the sampled random process. Part 1 of Theorem 4 suggests an approximation for the intensity, which we discuss next. Based on Part 1 of Theorem 4, without a formal proof, we use the approximation
\[
\lambda_\delta = E[N_\delta] \approx \frac{1}{\delta} \int_0^1 E\left[ \frac{|X'(t)|}{\max\{\alpha, |X(t)|\}} \right] dt
\]
assuming that the expectation on the RHS is finite. If \(\{X(t)\}\) is also a stationary process, then this reduces to
\[
\lambda_\delta \approx \frac{1}{\delta} E\left[ \frac{|X'(0)|}{\max\{\alpha, |X(0)|\}} \right].
\]
Further, since we are interested in $\alpha << 1$, we ignore $\alpha$, and in the following discussion, we focus on

$$\lambda_\delta \approx \frac{1}{\delta} E \left[ \frac{|X'(0)|}{|X(0)|} \right].$$

For example, suppose $\{X(t)\}$ is a non-negative lognormal process, that is, $\{\ln(X(t))\}$ is a stationary Gaussian process with zero mean and covariance function $R(t)$. Then the derivative of $\ln(X(t))$ equals $X'(t)/X(t)$ and it is also a stationary Gaussian process with zero mean and covariance function $-R''(t)$. Therefore, using the fact that a Gaussian random variable with zero mean and variance $\sigma$ has first absolute moment $\sqrt{2/\pi\sigma}$, we get,

$$\lambda_\delta \approx \frac{1}{\delta} E \left[ \frac{d\ln(X(t))}{dt} \right] = \frac{\sqrt{-2R''(0)}}{\sqrt{\pi}\delta}.$$  

For a random process bandlimited to $W$ Hz,

$$R(t) = A \frac{\sin(2\pi W t)}{2\pi W t}.$$
Fig. 3. The probability distribution function of $T(\delta; 0)$ when $\{\ln(X(t))\}$ is bandlimited process with $W = 1/2$. Our analytical approximation matches closely with that estimated by Monte-Carlo simulations.

and hence $-R''(0) = A(2\pi W)^2/3$. Thus

$$\lambda_\delta \approx \sqrt{\frac{8\pi A}{3}} \cdot \frac{W}{\delta},$$

that is, the intensity is linearly proportional to the bandwidth and inversely proportional to the Weber constant. In Figure 2, we demonstrate the accuracy of our analytical approximation for $\lambda_\delta$ by comparing with values obtained by Monte-Carlo simulation.

**Remark:** Shannon’s sampling theorem for bandlimited random processes states that from periodic samples spaced $1/2W$ apart, we can perfectly recover the random signal [30]. If $\sqrt{8\pi A/3}/\delta < 2$, then the Weber sampler has a smaller average sampling rate than the Nyquist rate $2W$, while otherwise, it has a higher average sampling rate. In the latter case, the higher number of samples is still justified in practice since the Weber sampler and linear interpolation lead to a small delay in reconstruction, while Shannon’s recovery formula needs infinite delay.

Parts 2) and 3) of Theorem 4 analyze the inter-sample times. Part 2) effectively shows that the
commonly sought after model of independent sample times is not valid for the Weber sampler. This is due to the fact that the future samples depend on the relative change with respect to the last sample, and the Markov chain \( \{X(T_n)\} \) carries information about the initial condition into the future. However, conditioned on this Markov chain, we do get independence of the inter-sample time. Part 3) studies the marginal and yields an approximation to the distribution of an inter-sample time. Let \( Z(t) := \max \left\{ \alpha, |X(t)| \right\} \) and \( F_{Z(t)}(z) \) denote its probability distribution function. Then Part 3) of Theorem 4 says that the probability distribution function of \( T(\delta; t) - t \) is approximated by

\[
F_{T(\delta; t)}(u) \approx F_{Z(t)} \left( \frac{u}{\delta} \right).
\]

Thus conditioned on \( T_{n-1} = t \),

\[
F_{T_n - T_{n-1}|T_{n-1}=t}(u|t) \approx F_{Z(t)} \left( \frac{u}{\delta} \right).
\]

In particular, if the process is stationary, then this conditional distribution does not depend on \( t \), and hence we get that the unconditional distribution of \( T_n - T_{n-1} \)

\[
F_{T_n - T_{n-1}}(u) \approx F_{Z(0)} \left( \frac{u}{\delta} \right).
\]

For example, if we assume that \( X(t) \) is lognormal and \( \{\ln(X(t))\} \) is a zero mean Gaussian process with covariance function \( R(t) \), then we get that

\[
Z(0) = \left| \frac{d \ln(X(t))}{dt} \right|_{t=0}.
\]

Since the derivative of a Gaussian process is also a Gaussian process, we get that \( Z(0) \) has the same distribution as the inverse of the absolute value of a Gaussian, that is, the probability distribution function

\[
F_{Z(0)}(u) = 2Q \left( \frac{1}{\sqrt{-R''(0)u}} \right),
\]

where the \( Q \)-function is defined as

\[
Q(v) := \int_v^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) dy.
\]

The corresponding probability density of \( Z_0 \) is given by

\[
\sqrt{-\frac{2}{\pi R''(0)}} \times \frac{1}{t^2} \exp \left( -\frac{1}{2R''(0)t^2} \right).
\]
In particular, we note that this density has a heavy tail of order $O(1/t^2)$, indicating that we may occasionally see very large inter-sample times. To demonstrate the accuracy of our approximation for the distribution function of $T(\delta; 0)$, in Figure 3, we compare the probability distribution function estimated from Monte-Carlo simulations with our analytical approximation for a bandlimited process with $W = 1/2$. We see that the approximation is quite good up to the 95 percentile level.

V. Conclusion

The Weber sampler attempts to sample a perceptual signal at the least rate without sacrificing on the quality of perception. In this paper, we proposed a regularized Weber sampler, and analyzed it for deterministic as well as random signals. Our class of signals is broad - allowing bandlimited signals, signals satisfying ODEs, signals with finite number of discontinuities and their random analogs. Our results show that good analytical approximation can be obtained for the sampling rate as well as the inter-sample times of the Weber sampler in the regime of practical interest - Weber constant $\delta < 0.2$. Our analytical results may be useful for analyzing the transmission of perceptual signals over communication networks, and currently we are pursuing this line of work. It is also of interest to extend the Weber sampler to vector sources (and even the exact nature of Weber’s law in such a setting needs more study).

VI. Proofs of Main Results

A. Proof of Theorem 1

Since the signal is differentiable, it is also continuous. Hence

$$\frac{|f(T_n) - f(T_{n-1})|}{\max\{\alpha, |f(T_{n-1})|\}} = \delta.$$  

Summing over $n$ and applying the Taylor expansion we get

$$\delta N_\delta = \sum_{n=1}^{N_\delta} \frac{|f(T_n) - f(T_{n-1})|}{\max\{\alpha, |f(T_{n-1})|\}} = \sum_{n=1}^{N_\delta} \frac{f'(\bar{T}_{n-1})(T_n - T_{n-1}) + f''(\bar{T}_{n-1})(T_n - T_{n-1})^2/2}{\max\{\alpha, |f(T_{n-1})|\}},$$

where $\bar{T}_{n-1} \in [T_{n-1}, T_n]$. Using

$$|x| - |y| \leq |x + y| \leq |x| + |y|,$$
we get that

\[ |S_1 - S_2| \leq \delta N_\delta \leq S_1 + S_2 \quad (4) \]

where

\[
S_1 := \sum_{n=1}^{N_\delta} \frac{|f'(T_{n-1})|(T_n - T_{n-1})}{\max\{\alpha, |f(T_{n-1})|\}} \quad (5)
\]

\[
S_2 := \sum_{n=1}^{N_\delta} \frac{|f''(\bar{T}_{n-1})|(T_n - T_{n-1})^2}{2 \max\{\alpha, |f(T_{n-1})|\}}. \quad (6)
\]

The following lemma, which is proved in Appendix I, gives us a handle on the inter-sample time \( T_n - T_{n-1} \).

*Lemma 1:* Under assumptions (A1), (A2), \( \lim_{\delta \to 0} (T_n - T_{n-1}) = 0 \), and there is a constant \( C \) that does not depend on \( \delta \) such that

\[ |f''(\bar{T}_{n-1})| (T_n - T_{n-1}) \leq C \delta^{1/3} \text{ for } \delta < 1. \]

Since \( \delta \to 0 \), \( T_n - T_{n-1} \to 0 \) for all \( n \), and we get that

\[ \lim_{\delta \to 0} S_1 = \int_0^1 \frac{|f'(t)|}{\max\{\alpha, |f(t)|\}} dt \]

which is the right hand side of (1). Thus to prove the result, from (4), we see that we need to show \( \lim_{\delta \to 0} S_2 = 0 \). To prove this we note that from Lemma 1,

\[ S_2 \leq C \delta^{1/3} \sum_{n=1}^{N_\delta} \frac{(T_n - T_{n-1})}{2 \max\{\alpha, |f(T_{n-1})|\}} \leq C \delta^{1/3} \frac{1}{2} \sum_{n=1}^{N_\delta} (T_n - T_{n-1}) \leq \frac{C \delta^{1/3}}{2} T_{N_\delta} \leq \frac{C \delta^{1/3}}{2}, \]

where we used \( T_{N_\delta} \leq 1 \). Thus \( S_2 \to 0 \) as \( \delta \to 0 \) and Part 1 of Theorem 1 follows.

The proof of Part 2 of Theorem 1 is related to that of Lemma 1. Hence the proof of this part is given in Appendix I along with the proof of Lemma 1 (see the Remark in Appendix I).

\[ \blacksquare \]

**B. Proof of Theorem 2**

To simplify notation, we give the proof when we have only one critical point; the proof for the case of more than one critical point follows similarly. Suppose the critical point occurs at time \( t_c \) and let \( \tau \) denote the first sampling instant of the Weber sampler after time \( t_c \), that is,

\[ \tau := \min \{ T_n : T_n \geq t_c \}. \]
Let $N_{\delta}(t_1, t_2)$ denote the number of samples taken by a Weber sampler in the interval $[t_1, t_2)$ starting with a sample at time $t_1$. Below, we shall show that

$$\lim_{\delta \to 0} \delta (N_{\delta}(\tau, 1) - N_{\delta}(t_c, 1)) = 0.$$  \hspace{1cm} (7)

Since

$$N_{\delta} = N_{\delta}(0, 1) = N_{\delta}(0, t_c) + N_{\delta}(\tau, 1),$$

we get that

$$\lim_{\delta \to 0} \delta N_{\delta} = \lim_{\delta \to 0} \delta N_{\delta}(0, t_c) + \lim_{\delta \to 0} \delta N_{\delta}(t_c, 1) + \lim_{\delta \to 0} \delta (N_{\delta}(\tau, 1) - N_{\delta}(t_c, 1)).$$

Using (7) and Theorem 1, the desired result follows.

It remains to show (7). Since $(\tau - t_c) \to 0$, (7) follows if we show that under the assumptions of Theorem 1,

$$\lim_{\delta \to 0} \delta |N_{\delta}(s_{\delta}, 1) - N_{\delta}| = 0, \text{ whenever } s_{\delta} \to 0.$$

We do so below. For simplicity of notation, we write $s_{\delta}$ as $s$. When we consider the first sample at $s$ instead of 0, we can repeat the same analysis as in the proof of Part 1 of Theorem 1. We recall that Part 1 of Theorem 1 relies on relating $\delta N_{\delta}$ to $S_1$ and $S_2$, and for $\delta N_{\delta}(s, 1)$, we similarly have terms $S_1(s)$, $S_2(s)$. Using the same bounding as in the proof of Theorem 1, we see that $S_2(s) \leq C \sqrt{\delta}$, where $C$ does not depend on $s$. Therefore, we get that

$$\lim_{\delta \to 0} \delta N_{\delta}(s, 1) = \lim_{\delta \to 0} \delta S_1(s)$$

provided these limits exist. However, using the fact that as $\delta \to 0$, the inter-sample times approach zero, and using the definition of Riemann integrals, we see that

$$\lim_{\delta \to 0} \delta S_1(s) = \int_0^1 \frac{|f'(t)|}{\max\{\alpha, |f(t)|\}} dt,$$

which is same as the result obtained by starting with a sample at 0. This completes the proof. \hfill \blacksquare

C. Proof of Theorem 3

By the Taylor expansion around $T_{n-1}$ as well as $T_n$ we have

$$f(t) = f(T_{n-1}) + (t - T_{n-1})f'(T_{n-1}) + (t - T_{n-1})^2 f''(\bar{T}_{n-1})/2$$

$$= f(T_n) + (t - T_n)f'(T_n) + (t - T_n)^2 f''(\bar{T}_n)/2.$$
Mixing these two representations in the same proportions as (3), we get,

\[ f(t) = \hat{f}(t) + \frac{(t-T_{n-1})(T_n-t)}{(T_n-T_{n-1})}(f'(T_{n-1}) - f'(T_n)) + \frac{(t-T_{n-1})^2(T_n-t)}{2(T_n-T_{n-1})}f''(T_{n-1}) + \frac{(t-T_{n-1})(T_n-t)^2}{2(T_n-T_{n-1})}f''(T_n). \]

It is easy to check that the maximum of \((t-T_{n-1})(T_n-t)\) is \((T_n-T_{n-1})^2/4\). Using \(|f''(t)| \leq L\), we get that

\[ |\hat{f}(t) - f(t)| \leq \frac{(f'(T_{n-1}) - f'(T_n))}{4(T_n-T_{n-1})}(T_n-T_{n-1})^2 + \frac{L}{4}(T_n-T_{n-1})^2. \]

Since \(f'\) has a continuous derivative, it is also Lipschitz with constant bounded by \(L\) and the fraction in the first term is bounded. The desired bound on the absolute error now follows.

To prove the bound on the supremum of \(e(\hat{\delta}; t)\), we note that under the additional assumption that \(|f'(t)| \geq \epsilon > 0\), from the proof of Part 2 of Theorem 1 in Appendix I, \(T_n-T_{n-1} \leq C' \delta\) for some constant \(C'\).

\[ \text{APPENDIX I} \]

\[ \text{PROOF OF LEMMA 1} \]

Let \(t \in [0, 1]\) and define

\[ T(\delta; t) := \inf \left\{ u \in (t, 1] : \frac{|f(u) - f(t)|}{\max\{\alpha, |f(t)|\}} > \delta \right\}. \]

We note that \(T(\delta; T_{n-1}) = T_n\). By continuity of \(f(t)\), it follows that

\[ \frac{f(T(\delta; t)) - f(t)}{\max\{\alpha, |f(t)|\}} = \pm \delta. \]

Using the Taylor expansion and for simplicity denoting \(T(\delta; t) - t\) by \(u\), we get,

\[ f'(t)u + f''(\bar{t})u^2/2 = \pm \max\{\alpha, |f(t)|\} \delta, \tag{8} \]

where \(\bar{t} \in [t, T(\delta; t)]\). The solutions to (8) are

\[ u = \frac{-f'(t) \pm \sqrt{(f'(t))^2 + 2f''(t) \max\{\alpha, |f(t)|\} \delta}}{f''(t)}. \tag{9} \]

Our first aim is to show that any positive root of (8) satisfies \(\lim_{\delta \to 0} u = 0\) (which implies that \(T_n-T_{n-1} \to 0\)). To this end we consider different cases. When \(f'(t) \neq 0\), and \(f(t)\) is linear in \([t, T(\delta; t)]\), (8) is a linear equation and the result follows immediately. Hence assume...
that $f''(s) \neq 0$ in a neighborhood of $t$. Then by using (9), we can verify that the positive root satisfies
\[
\lim_{\delta \to 0} \frac{u}{\delta} = \frac{\max\{\alpha, |f(t)|\}}{|f'(t)|},
\]
and hence $u \to 0$. Consider next the case $f'(t) = 0$. By assumption (A2), we see that $f''(t) \neq 0$, and by assumption (A2) we see that the second derivative is non-zero in a neighborhood of $t$. Therefore $|f''(t)| > \epsilon > 0$ for this small neighborhood and from (8) we get that
\[
\epsilon u^2/2 \leq \max\{\alpha, |f(t)|\} \delta,
\]
which implies that $u \to 0$.

**Remark:** We note that (10) also established the first limit in Part 2 of Theorem 1. To prove the second limit, we only need to substitute $f'(t) = 0$ in the expression for $u$, use the fact that $\bar{t} \to t$ as $\delta \to 0$, and the continuity of $f''(t)$

Next we establish the bound on $|f''(\bar{t})|u$. From (9), if $f'(t) = 0$, then
\[
|f''(\bar{t})|u = \sqrt{2|f''(\bar{t})| \max\{\alpha, |f(t)|\}} \delta.
\]
Since the signal and its second derivative are continuous on $[0,1]$, they are also bounded, and we get that $|f''(\bar{t})|u \leq C\sqrt{\delta} \leq C\delta^{1/3}$ for $\delta < 1$. Next consider the remaining case of $f'(t) \neq 0$. We derive the bound for the case $f'(t) > 0$ first; the result for $f'(t) < 0$ is treated separately below. In this case, for the positive root of (8),
\[
|f''(\bar{t})|u = -f'(t) + \sqrt{(f'(t))^2 + 2|f''(t)| \max\{\alpha, |f(t)|\}} \delta.
\]
But for non-negative numbers $a, b$, we have the following inequality for $\delta < 1$ (which is proved below)
\[
\sqrt{a + b\delta} \leq \sqrt{a} + \left[\frac{\sqrt{a}}{2} + \sqrt{2b}\right] \delta^{1/3}.
\]
(11)
Applying this bound, we get
\[
|f''(\bar{t})|u \leq \left[\frac{f'(t)}{2} + 2\sqrt{|f''(\bar{t})| \max\{\alpha, |f(t)|\}}\right] \delta^{1/3}
\]
and once again since the functions on the RHS are bounded, the result $|f''(\bar{t})|u \leq C\delta^{1/3}$ follows.

It only remains to show (11). Towards this end we note that for $a < b\delta^{2/3}$,
\[
\sqrt{a + b\delta} < \sqrt{b\delta^{2/3} + b\delta} < \sqrt{2b\delta^{1/3}} \text{ since } \delta < 1.
\]
On the other hand, for \( a \geq b\delta^{2/3} \), using \( \sqrt{1 + x} \leq 1 + x/2 \),

\[
\sqrt{a + b\delta} = \sqrt{a} \sqrt{1 + \frac{b\delta}{a}} \leq \sqrt{a} \left( 1 + \frac{b\delta}{2a} \right) \leq \sqrt{a} \left( 1 + \frac{\delta^{1/3}}{2} \right).
\]

Since both these bounds are positive, their sum bounds \( \sqrt{a + b\delta} \) for all values of \( a, b\delta \), and this gives the bound (11).

Finally, consider the case \( f'(t) < 0 \). Since we are interested in the limit \( \delta \to 0 \), we consider the case

\[
|f'(t)|^2 \geq 2f''(\bar{t}) \max\{\alpha, |f(t)|\} \delta.
\]  

(12)

Under this condition, the smallest positive root from (9) satisfies

\[
|f''(\bar{t})|u = -f'(t) - \sqrt{(f'(t))^2 - 2|f''(t)| \max\{\alpha, |f(t)|\} \delta}
\]

\[
= |f'(t)| \left[ 1 - \sqrt{1 - \frac{2|f''(\bar{t})| \max\{\alpha, |f(t)|\} \delta}{(f'(t))^2}} \right]
\]

\[
\leq |f'(t)| \left[ 1 - 1 - \frac{2|f''(\bar{t})| \max\{\alpha, |f(t)|\} \delta}{(f'(t))^2} \right]
\]

\[
= \frac{2|f''(\bar{t})| \max\{\alpha, |f(t)|\} \delta}{|f'(t)|},
\]

where we used the inequality \( \sqrt{1 - x} \geq 1 - x \). Further applying (12), we get

\[
|f''(\bar{t})|u \leq \sqrt{2|f''(t)| \max\{\alpha, |f(t)|\} \delta} \leq C\delta^{1/3}.
\]

This completes the proof of Lemma 1.

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