Poset Dimension and Boxicity

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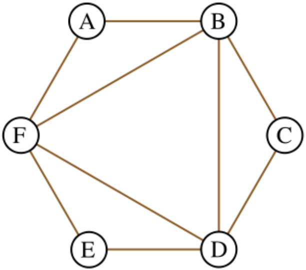
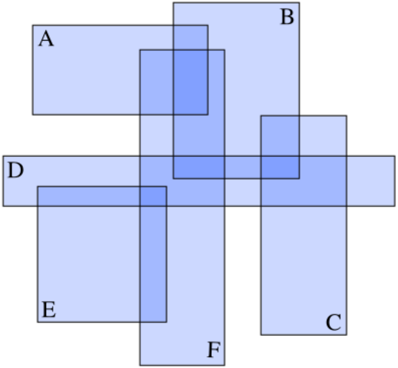
Boxicity (a geometric definition)

- A k-box is a Cartesian product of k intervals.
- The boxicity of a graph G, $box(G)$ is the minimum integer k such that G can be represented as an intersection graph of k-boxes in the k-dimensional Euclidean space.





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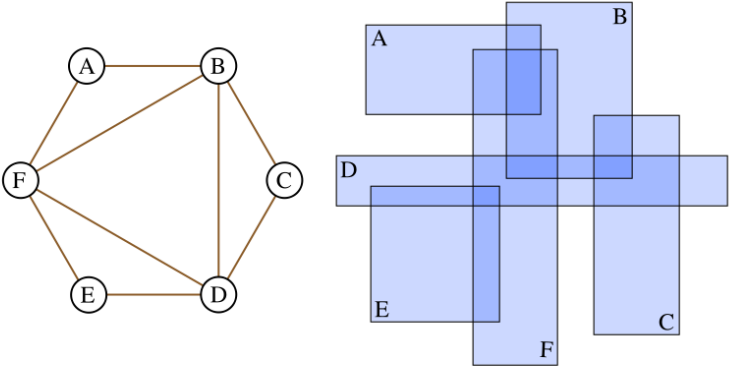

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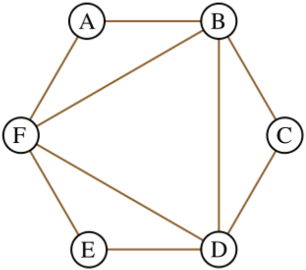

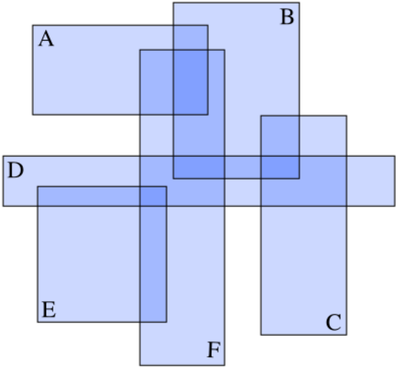

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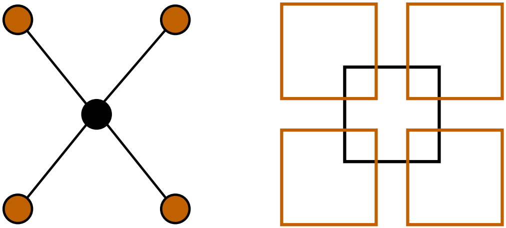
Cubicity

The cubicity of a graph is the minimum dimension k in which a given graph can be represented as an intersection graph of axis parallel k-dimensional unit cubes.



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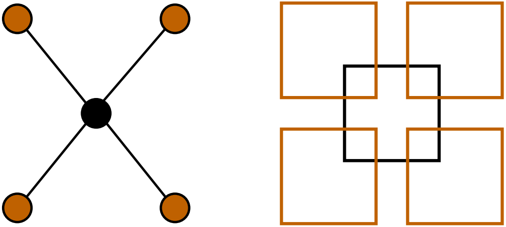

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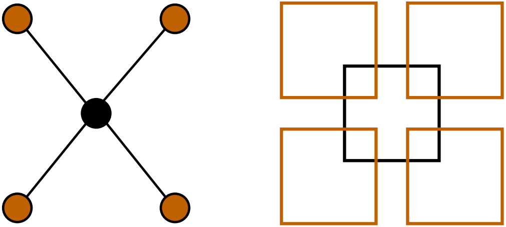

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- Boxicity and cubicity were introduced by Roberts in 1969.
- Determining if the boxicity of a graph is ≥ 2 is an NP-complete problem [Kratochvíl].
- Boxicity of a graph is at most $\lfloor n/2 \rfloor$ where n is the order of the graph.
- Determining if the cubicity of a graph is ≥ 3 is an NP-complete problem [Yannakakis].
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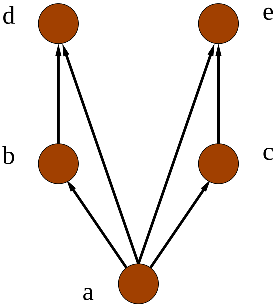
Partially Ordered Set (Poset)

$\mathcal{P} = (X, P)$ where X is a non-empty finite set and P is the partial order on X satisfying reflexivity, anti symmetry and transitivity.



$a \longrightarrow b$ implies $a < b$
$X = \{a, b, c, d, e\}$
$P =$
$\{(a, b), (a, c), (a, d), (a, e), (b, d), (c, e)\}$

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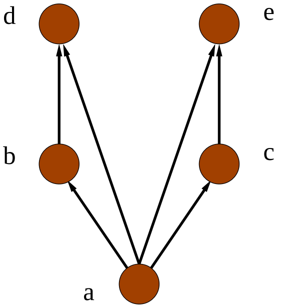

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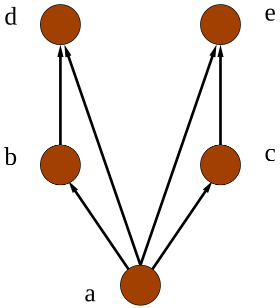
Dimension of a Poset

The dimension of a poset $\mathcal{P}(X, P)$, denoted by $\dim(\mathcal{P})$ is the minimum integer k such that there exist k total orders $\{L_1, L_2, \ldots, L_k\}$ satisfying: for any two distinct elements x and y, $x < y$ in P if and only if $x < y$ in each L_i.



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- Introduced by Dushnik and Miller in 1941
- The dimension of a poset is 1 if and only if it is a linear order.
- It is NP-complete to determine if the dimension of a poset is at most 3 [Yannakakis].
- Hegde and Jain proved that there exists no polynomial-time algorithm to approximate the dimension of an n-element poset within a factor of $O(n^{0.5-\epsilon})$ for any $\epsilon > 0$, unless $NP = ZPP$.

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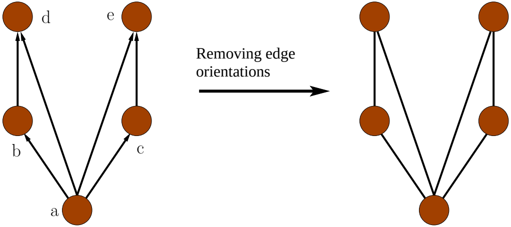
Posets and Comparability Graphs

The underlying comparability graph of a poset $\mathcal{P} = (X, P)$ is an undirected simple graph with vertex set X and two vertices are adjacent in G if and only if they are comparable in $\mathcal{P}$. We will denote it by $G_{\mathcal{P}}$.



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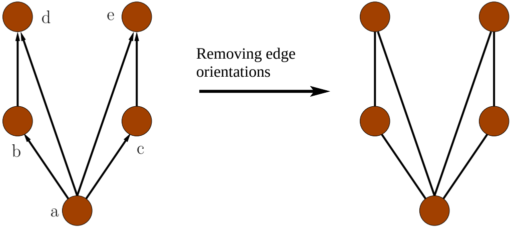

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First Result

Let $\mathcal{P}$ be a poset and $G_{\mathcal{P}}$ its underlying comparability graph. Then, $\mathrm{box}(G_{\mathcal{P}})/(\chi(G_{\mathcal{P}}) - 1) \leq \dim(\mathcal{P}) \leq 2\mathrm{box}(G_{\mathcal{P}})$, where, $\chi(G_{\mathcal{P}})$ is the chromatic number of $G_{\mathcal{P}}$.

Height-2 Posets

For a height-2 poset, the underlying comparability graph is a bipartite graph. Therefore,

$$\mathrm{box}(G_{\mathcal{P}}) \leq \dim(\mathcal{P}) \leq 2\mathrm{box}(G_{\mathcal{P}}),$$

Some tight examples:

- a 4-cycle: $\mathrm{box}(G_{\mathcal{P}}) = \dim(\mathcal{P}) = 2$.
- The crown poset S_n^0: $\mathrm{box}(G_{\mathcal{P}}) = n/4$ while $\dim(\mathcal{P}) = n/2$.

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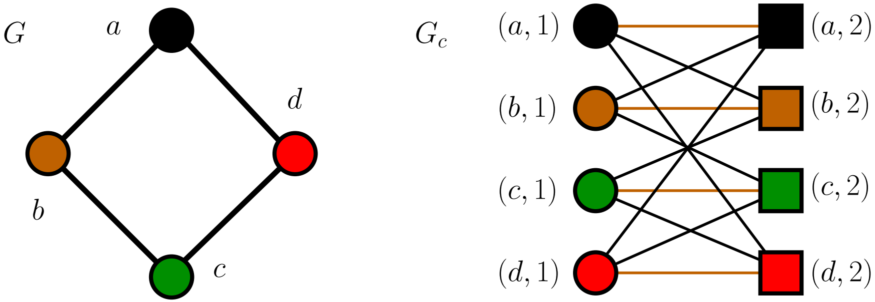
Second Result

Extended Double Cover



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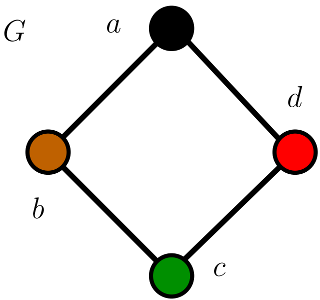

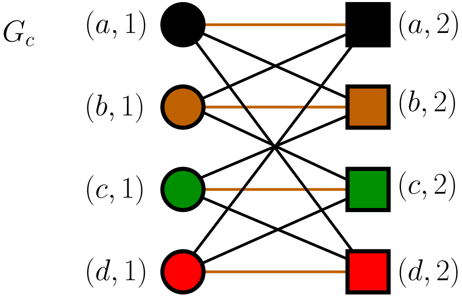

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On combining the two results...

We can associate with every graph G, a height-2 poset $\mathcal{P}_C$ such that $\text{box}(G) = \Theta(\dim(\mathcal{P}_C))$.

The constant involved here is between $1/2$ and 2.

Some Interesting Consequences

New upper bounds for poset dimension:

- $\dim(\mathcal{P}) \leq 2\,\text{tree-width}\,(G_{\mathcal{P}}) + 4$ (from [Chandran and Sivadasan])
- $\dim(\mathcal{P}) \leq \text{ MVC}\,(G_{\mathcal{P}}) + 2$ (from [Chandran, Das and Shah])
- $\dim(\mathcal{P}) \leq 4$ if $G_{\mathcal{P}}$ is outer planar and $\dim(\mathcal{P}) \leq 6$ if $G_{\mathcal{P}}$ is planar (from [Scheinerman] and [Thomassen] respectively)

Some Interesting Consequences...

Boxicity and maximum degree (Δ):

- $\text{box}(G) < c\Delta(\log \Delta)^2$, where c is some constant (from a result of [Füredi and Kahn]). This is an improvement over the previously best known bound of $\Delta^2 + 2$ [Esperet].
- There exist graphs with boxicity $\Omega(\Delta \log \Delta)$ (from a result of [Erdős, Kierstead and Trotter]). This disproves a conjecture by Chandran et al. that $\text{box}(G) = O(\Delta)$.

There exists no polynomial-time algorithm to approximate the boxicity of a bipartite graph on n-vertices with a factor of $O(n^{0.5-\epsilon})$ for any $\epsilon > 0$, unless $NP = ZPP$. (from [Hegde and Jain])

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