Introduction to Property Testing

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Outline

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One Main Goal: Have running time polynomial.

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Property Testing

In Property testing we are usually interested in sub-linear query complexity, that is, we want to read a small fraction of the input.

But how is it possible?

We have to give up on something. We will assume some promise on the input.

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The goal is to answer it in CONSTANT time and hence can't even read the whole input. $--$ Not Possible

But, say, there is a promise that either $x = y$ OR x and y differ at more than $1/4$ fraction of the indices. Then ...

Simple sampling algorithm for testing of equality

Algorithm

Randomly pick 4 indices $\{i_1, i_2, i_3, i_4\}$ uniformly and independently at random. If

$$x_{i_1} = y_{i_1}, x_{i_2} = y_{i_2}, x_{i_3} = y_{i_3}, x_{i_4} = y_{i_4},$$

then ACCEPT otherwise REJECT.

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- If $x = y$ then the algorithm always ACCEPTS.
- If x and y differ at $1/4$ fraction of the indices then the algorithm ACCEPTS with probability at most $1/3$.

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If the goal is to sample a small part of the voters then its not possible always to give the right answer (even with high probability).

But if we want to distinguish between whether Party A wins by a big margin or Party B wins by a big margin: then statistical sample works.

Example: Checking Bipartiteness.

2-colorability

Given an undirected graph G can we color the vertices of G with 2 colors such that no adjacent vertices are of the same color? Or in other words is it bipartite.

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In general it may require us to look at the whole graph to answer but can we look at a very small fraction of the graph and distinguish

- The graph is bipartite
- A "lot" of edges have to be removed to make it bipartite.

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Promise Problem

For a property $\mathcal{P}$ and a distance parameter ϵ, given an input x distinguish between the two cases:
(a) Is $x \in \mathcal{P}$, OR (b) Is x ϵ-far from $\mathcal{P}$.

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- **Branching Program Testing:** Given a truth-table of a function f test is the function is accepted by a constant depth read-once branching program OR is far from being accepted by a constant depth read-once branching program.

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- **Branching Program Testing:** Given a truth-table of a function f test is the function is accepted by a constant depth read-once branching program OR is far from being accepted by a constant depth read-once branching program.
- **Isomorphism Testing:** Given two objects $\mathcal{O}_1$ and $\mathcal{O}_2$ test are the two isomorphic OR far-from being isomorphic. (For example: Graph Isomorphism or Function Isomorphism)

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In the rest of the talk we would not consider the running time of an algorithm but rather the number of bits of the input that is read. Accessing each bit of the input is called a QUERY.

Property tester

Definition

Let $\mathcal{P}$ be a property. A tester for $\mathcal{P}$ is a *randomized* algorithm $\mathcal{A}$ with black box access to an input x and satisfies:

- If $x \in \mathcal{P} \Rightarrow \Pr[\mathcal{A} \text{ accepts}] \geq 2/3$.
- If x is ϵ-far from $\mathcal{P} \Rightarrow \Pr[\mathcal{A} \text{ rejects}] \geq 2/3$.

We allow the algorithm to be *adaptive* (queries may depend on the outcome of previous queries).

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Trivial example: let $\mathcal{P}$ be the property "$x \equiv 0$". Then taking $O(1/\epsilon)$ independent samples works w.h.p.

Other connected areas...

- Statistical estimation - In property testing we consider more combinatorial objects like properties of Boolean functions and graphs. .
- Evasiveness and Certificate Complexity.
- Probabilistically Checkable Proofs (PCP).
- Locally Decodable Codes.

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- How the input is represented? For example, is the graph given as adjacency matrix or adjacency list or some other way. [Dense graph model, sparse graph model, orientation model in graph testing]
- How the queries are made? [Classical, quantum]
- Do we also want to accept inputs that are "close" to the property? [Tolerant model and Intolerant Model]

What kind of questions to ask?

- Given a property $\mathcal{P}$ what is the query complexity for testing $\mathcal{P}$.
  - Design a property tester that tests $\mathcal{P}$ using $O(q)$ number of queries.
  - Prove that no property tester can test using less than $\Omega(q)$ number of queries.

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  - Prove that no property tester can test using less than $\Omega(q)$ number of queries.
- Classify the set of properties that can be tested using constant number of queries.
- Come up with the right model for testing.

Outline

1-sided error testers

1-sided-error property tester

Let $\mathcal{P}$ be a property. A 1-sided-error property tester for $\mathcal{P}$ is a *randomized* algorithm $\mathcal{A}$ with black box access to an input x and satisfies:

- (Completeness) If $x \in \mathcal{P} \Rightarrow \Pr[\mathcal{A} \text{ accepts}] = 1$.
- (Soundness) If x is ϵ-far from $\mathcal{P} \Rightarrow \Pr[\mathcal{A} \text{ rejects}] \geq 2/3$.

We allow the algorithm to be *adaptive* (queries may depend on the outcome of previous queries).

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- The tester has to ACCEPT if the input satisfies the property.
- Hence, the only way the tester can reject is if it find a PROOF that the input does not satisfy the property.
- So if the input does not have the property then the tester must find a PROOF/WITNESS with high probability.

Typical 1-sided-error tester

1-sided-error algorithm

Query some bits of the input. The bits to be queried can be either uniformly chosen or chosen in a cleaver co-related fashion.

- If the answers of the queried bits contains a WITNESS that the input is not in the property then REJECT
- Else ACCEPT

Goal is to use some nice structure for the property for making the queries, like

- the Szemeredi's Regularity Lemma for graphs,
- properties of Fourier coefficients for algebraic functions, etc

Usually, the proof of SOUNDNESS is the hard part.

So what is the success probability of the tester?

Say the tester uses the random string r and queries the bits in Q_r (also say $|Q_r| = q$). Then the probability of success is

$$\Pr_r[Q_r \text{ contains a WITNESS}].$$

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$$\Pr_r[Q_r \text{ contains a WITNESS}].$$

Thus a 1-sided-error property tester can successfully test a property $\mathcal{P}$ with q queries only if, an input x is "far" from $\mathcal{P}$ implies there is a lots of WITNESS of size q hidden in x.

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For example: Checking whether $f : [n] \to [n]$ is 1-to-1 or 2-to-1 requires at least $\sqrt{n}$ queries. (By Birthday Paradox)

2-sided-error property tester

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Let $\mathcal{P}$ be a property. A 2-sided-error tester for $\mathcal{P}$ is a *randomized* algorithm $\mathcal{A}$ with black box access to an input x and satisfies:

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- The tester can use estimation/approximation as a tool.

For example: Distinguishing whether a string $x \in \{0,1\}^n$ has $n/4$ $1's$ OR $n/3$ $1's$ can be done using CONSTANT number of queries.

In general 2-sided-error algorithms can be very complicated.

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And now if one shows that any deterministic algorithms that makes q queries cannot distinguish the two kind of inputs then by Yao's Lemma we obtain a lower bound of q on the query complexity for testing $\mathcal{P}$.

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And now if one shows that any deterministic algorithms that makes q queries cannot distinguish the two kind of inputs then by Yao's Lemma we obtain a lower bound of q on the query complexity for testing $\mathcal{P}$.

So, if the distribution of answers to the queries are similar when the input is drawn according to $\mathcal{D}_N$ and when it is drawn according to $\mathcal{D}_Y$ then the query complexity is $\geq q$.

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- Queries are of form: $x \in \Sigma^n \longrightarrow f(x)$.

Property Tester for $\mathcal{P}$

A 1-sided-error tester for $\mathcal{P}$ is a *randomized* algorithm $\mathcal{A}$ that given query access to a truth-table of a function f does the following:

- If $f \in \mathcal{P} \Rightarrow \Pr[\mathcal{A} \text{ accepts}] = 1$.
- If for at least $\epsilon|\Sigma|^n$ number of strings in Σ^n the value of f has to be changed so that the property $\mathcal{P}$ is satisfied then $\Pr[\mathcal{A} \text{ rejects}] \geq 2/3$.

Testing of Linearity

Linearity testing

Given query access to a Boolean function $f : \{0,1\}^n \to \{0,1\}$ test if f is linear, that is, if for all $x, y \in \{0,1\}^n$, $f(x) \oplus f(y) = f(x \oplus y)$.

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The obvious test is the following: pick two random $x, y \in \{0,1\}^n$ and if $f(x) \oplus f(y) \neq f(x \oplus y)$ then REJECT else ACCEPT.

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Linearity Testing [Blum-Luby-Rubinfeld]

The above tester has the following properties:

- If f is linear then the tester always ACCEPTS.
- If f is ϵ-far from linear then the tester REJECTS with high probability. (Proof using Fourier Analysis).

Generalization of Linearity Testing

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Low-degree testing [Babai-Fortnow-Lund, Rubinfeld-Sudan]

The query complexity for testing degree d polynomials is a function of $|\mathbb{F}|$ and d. When $|\mathbb{F}| = 2$ then the query complexity is 2^d and when $|\mathbb{F}|$ is around d then the query complexity is $poly(d)$.

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This tester in also used in Probabilistically Checkable Proofs (PCP) [Arora-Safra, Arora-Lund-Motwani-Sudan-Szegedy]

Degree d tester, when $\mathbb{F} > d$.

Algorithm (For $|\mathbb{F}| > d$)

- *Pick a random* $x \in \mathbb{F}^n$
- *Pick a random line through* x. *Pick a random* $y \in \mathbb{F}^n$ *and consider all points of form* $x + \lambda y$.
- *Query at all the* $|\mathbb{F}|$ *points.*
- *If* f *is a degree* d *polynomial then restricted to this line it is a degree* d *univariate polynomial in variable* λ.
- *Use the points* $f(x + \lambda y)$, *when* $\lambda \neq 0$ *to fit a degree* d *polynomial.*
- *If the polynomial evaluated at* $\lambda = 0$ *is equal to* $f(x)$ *then ACCEPT else REJECT.*

Monotonicity Testing

Given query access to a function $f : \{0,1\}^n \to \mathbb{R}$ test if f is monotone, that is, if $x, y \in \{0,1\}^n$ are such that for all $i \in [n]$ $x_i \leq y_i$ then $f(x) \leq f(y)$.

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Monotonicity Testing [Chakrabarty-Seshadhri, Briet-C-Garcia-Soriano-Matsliah]

The 1-sided-error query complexity for testing monotonicity with arbitrary range is $\Theta(n)$.

Monotonicity Testing

Given query access to a function $f : \{0,1\}^n \to \mathbb{R}$ test if f is monotone, that is, if $x, y \in \{0,1\}^n$ are such that for all $i \in [n]$ $x_i \leq y_i$ then $f(x) \leq f(y)$.

Monotonicity Testing [Chakrabarty-Seshadhri, Briet-C-Garcia-Soriano-Matsliah]

The 1-sided-error query complexity for testing monotonicity with arbitrary range is $\Theta(n)$.

The upper bound is just a pair-tester where the tester picks $x \in \{0,1\}^n$ and an $i \in \{1, \ldots, n\}$ at random and checks if $f(x)$ and $f(x \oplus e_i)$ satisfies the monotonicity property.

Repeat it $O(n)$ times.

Testing of combinatorial/complexity measures of functions.

Testing of BP [Newman]

Given query access to a function $f : \{0,1\}^n \to \{0,1\}$, testing if f is accepted by a width w read-once branching program can be done using $O(w)$ number of queries.

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Testing of Circuit size [Diakonikolas-Lee-Matulef-Onak-Rubinfeld-Servedio, C-Garcia-Soriano-Matsliah]

Given query access to a function $f : \{0,1\}^n \to \{0,1\}$, testing if f is accepted by a circuit of size s has query complexity $s^{\Theta(1)}$.

Properties of Distributions

A $f : \{1, \ldots, n\} \to \{1, \ldots, k\}$ defines a distribution $\mathcal{D}_f$ on $\{1, \ldots, k\}$, where

$$\Pr_{x \leftarrow \mathcal{D}_f}[x = i] = |f^{-1}(i)|/n.$$

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2-side-error query complexity for testing uniformity is $\tilde{\Theta}(\sqrt{k})$.

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Lower bound: Distinguishing whether f is uniform with support size k from f is uniform with support size $k/2$ requires $\sqrt{k}$ queries. Just like distinguishing 1-to-1 function from 2-to-1 functions. $\square$

Outline

Testing of graph property

- A property $\mathcal{P}$ is a set of graphs. For example: all bipartite graphs, all graphs that is isomorphic to a particular graph, all graphs where there exists a path from vertex 1 to vertex 2, ...

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Definition

A 1-sided-error tester for $\mathcal{P}$ is a *randomized* algorithm $\mathcal{A}$ that given query access to a graph G does the following:

- If $G \in \mathcal{P} \Rightarrow \Pr[\mathcal{A} \text{ accepts}] = 1$.
- If at least $\epsilon\binom{|V|}{2}$ number of entries of the adjacency matrix has to be changed so that the property $\mathcal{P}$ is satisfied then $\Pr[\mathcal{A} \text{ rejects}] \geq 2/3$.

Testing of Bipartiteness in the dense graph model

Given query access to the adjacency matrix of a graph G, test if G is bipartite of one has to remove $\epsilon\binom{|V|}{2}$ edges to make it bipartite.

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Algorithm [Goldreich-Goldwasser-Ron]

- Pick $O(1/\epsilon^2 \log(1/\epsilon))$ number of vertices at random.
- Query all the pairs of selected vertices.
- If the induced graph is not bipartite REJECT else ACCEPT

Proof: If the graph is bipartite the algorithm always accept.

So now we have to prove that if G is ϵ-far from being bipartite then the induced graph is not bipartite with high probability.

Proof of Soundness of the Algorithm for Testing Bipartiteness

Since it is a 1-sided-error algorithm for every possible bipartition of the vertex set we should catch a violating edge, that is edges within the same part.

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Note that given a particular bipartition by randomly sampling of $O(1/\epsilon^2)$ edges we would catch a violation for that bipartition with high probability. But we have to catch for all the bipartitions with high probability. Unfortunately, simple union bound does not give the math as the number of such bipartitions is $2^{|V|}$.

Proof of Soundness of the Algorithm for Testing Bipartiteness (contd...)

So we think of the selected vertices as two sets V_A and V_B. Vertices V_A induces the subgraph G_A.

After we have queried the subgraph G_A we show only a "small" number of partitions survive with high probability.

And then we can say, using union bound, that the second set V_B helps to catch the violations for the small number of surviving bipartitions.

Various other Graph Properties

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- Is there a clique of size ρn $--$ $O(1/\epsilon)$ number of queries. (2-sided-error)
- Triangle free-ness $--$ $tower(1/\epsilon)$. (Using Regularity Lemma)

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The proof roughly based on the idea that testing monotone graph properties can be reduced to testing whether the graph has a regular-partition with certain parameters.
And testing whether a graph has a regular-partition can be tested with constant number of queries.

In the dense-graph-model its all about regularity

Theorem (Alon-Fischer-Newman-Shapira)

A graph property P can be tested with a constant number of queries if and only if testing P can be reduced to testing the property of satisfying one of finitely many Szemeredi-partitions.

Testing of Connectivity

Problem

Can we test whether in a graph is connected?

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All graphs are just $|V|$ changes away from being connected and hence all graphs are ϵ-close to being connected.

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So we need some other models for sparse-graph-properties.

Sparse Graph Model

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Property Tester for Bounded Degree Model

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- If $G \in \mathcal{P} \Rightarrow \Pr[\mathcal{A} \text{ accepts}] = 1$.
- If at least ϵm number of edges has to be added or removed so that the property $\mathcal{P}$ is satisfied then $\Pr[\mathcal{A} \text{ rejects}] \geq 2/3$.

Testing Connectivity in Sparse Graph Model

Observation

If a graph G is ϵ-far (in the sparse-graph-model) from being connected then it has more than $\epsilon m + 1$ connected components. And thus it must have at least $(\epsilon/2)m$ number of components of size at most $2n/\epsilon m$.

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Algorithm

- *Randomly pick $4n/\epsilon m$ vertices.*

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- *Do a BFS from each of the selected vertices till you find $2n/\epsilon m$ vertices.*

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Algorithm

- *Randomly pick $4n/\epsilon m$ vertices.*
- *Do a BFS from each of the selected vertices till you find $2n/\epsilon m$ vertices.*
- *If you find a component of size less than $2n/\epsilon m$ then REJECT, else ACCEPT.*

Other problems that has constant query complexity in the sparse graph model

- Cycle-freeness,
- Eulerianess,
- subgraph freeness

All the above has similar algorithms to connectivity testing.

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Problem

Can we test whether in a graph there is a path from a given vertex s to another given vertex t?

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Property Tester for Orientation Model

A 1-sided-error tester for $\mathcal{P}$ is a *randomized* algorithm $\mathcal{A}$ that given query access to a graph G does the following:

- If $G \in \mathcal{P} \Rightarrow \Pr[\mathcal{A} \text{ accepts}] = 1$.
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Testing in the orientation model

st-connectivity [C-Fischer-Lachish-Matsliah-Newman]

There is a 1-sided-error tester that makes $2^{2^{2^{O(1/\epsilon)}}}$ number of queries and tests for st-connectivity in the orientation model (ϵ is the distance parameter).

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There is a 1-sided-error tester that makes $2^{2^{2^{O(1/\epsilon)}}}$ number of queries and tests for st-connectivity in the orientation model (ϵ is the distance parameter).

- Other properties like Eulerianness has also been studied in this model. But their query complexity is not constant.
- Not many properties are known to have constant query complexity in the orientation model.
- Even proving that a constant size witness exist is also hard. For example: If G is ϵ far from being s-to-all connected then does there exist a constant size witness?

Characterization in the Sparse-Graph-Model and Orientation-Model

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Characterization of properties that can be tested using constant number of queries in the Sparse-Graph-Model. −− OPEN

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- Function Property Testing
  - Linearity, low degree, constant-width-read-once-BP, k-juntas have constant query complexity
  - Monotonicity - query complexity is $\Omega(n)$ and $O(n^2)$
  - Testing distribution - uniformity testing has query complexity $\tilde{\Theta}(\sqrt{|Range|})$.

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  - Dense Graph Model - Testing is all about regularity,
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Current directions....

- Get tight query complexity for testing various properties.
- Classify Boolean function properties that can be tested using constant number of queries.
- Connection to communication complexity: like connection to gap-Hamming problem.
- Get lower bounds on the query complexity (dependence on ϵ) for graph properties: connection to additive combinatorics.
- Connection to LDC/PIR.
- Connection to learning theory.

Outline

Testing of Graph Isomorphism

Let H be a fixed graph. Then given query access to the adjacency matrix of a graph G test if G is isomorphic to H or if G is ϵ-far from being isomorphic to H.

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GI Testing with constant number of queries [Fischer]

The query complexity for testing isomorphism to a fixed graph is constant iff the given graph is close to a graph that is generated by a constant number of cliques.

Hyper-graph isomorphism testing and its generalizations

Hyper-Graph Isomorphism testing: Let H be a fixed d-refular-hypergraph. Then given query access to the adjacency matrix of a d-regular-hypergraph G test if G is isomorphic to H or if G is ϵ-far from being isomorphic to H.

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Testing Isomorphism under Group Operations: Let $\mathcal{G}$ be a primitive subgroup of S_n. Let $x \in \{0,1\}^n$ be a fixed string. Then given a string $y \in \{0,1\}$ test if x is isomorphic to y under permutation of the indices by elements of the group $\mathcal{G}$, that is, is there a $\pi \in \mathcal{G}$ such that for all i, $x_i = y_{\pi(i)}$, OR for all $\pi \in (G)$ for at least ϵn indices i, $x_i \neq y_{\pi(i)}$.

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Testing Isomorphism under Group Operations [Babai-C]

The query complexity for test isomorphism under primitive group operation is $\tilde{\Theta}(\log |G|)$. This implies the query complexity for testing d-regular hypergraph isomorphism is $\tilde{\Theta}(|V|)$.
For 2-sided-error the bounds are $\tilde{\Theta}(\sqrt{\log |G|})$ and $\tilde{\Theta}(\sqrt{|V|})$ respectively.

Boolean Function Isomorphism Testing

Let $f : \{0,1\}^n \to \{0,1\}$ be a fixed function. Then given query access to the truth-table of a function g test if g is isomorphic to f upto a permutation of its variable, that is, does there exist a permutation $\pi \in S_n$ such that for all x, $f(x^\pi) = g(x)$, where $x_i^\pi = x_{\pi(i)}$.

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For example:

- Is the function g a dictator function? $--$ Constant query complexity.
- Is the function a parity on k variable? $--$ Query complexity $O(k \log k)$ and $\Omega(k)$
- Is the function isomorphic to Majority? $--$ Constant Query Complexity.

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Boolean FI testing [Alon-Blais, C-Garcia-Soriano-Matsliah]

The 1-sided-error query complexity for testing isomorphism to a k-junta is $\Theta(k \log n)$ where as the 2-sided-error query complexity is $O(k \log k)$.

Characterization?

Just like in case of graph isomorphism Fischer proved that query complexity is constant iff the graph is generated by a constant number of cliques, can we say something like that for function isomorphism.

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Conjecture

If $f(x)$ depends on $|X|$ and at most k indices then the query complexity for testing isomorphism to f is $O(k \log k)$ and $\Omega(\log k)$.

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