The Learning with Rounding Problem: Reductions and Applications

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(Thanks: Chris Peikert)

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August 15, 2013

Pseudorandom Functions [GGM'84]

- A family $\mathcal{F} = \{F_s : \{0,1\}^k \to D\}$ s.t. given adaptive query access,

$$\boxed{F_s \leftarrow \mathcal{F}} \qquad \overset{c}{\approx} \qquad \boxed{\text{random func } U}$$

$$\uparrow x_i \quad \downarrow F_s(x_i) \qquad\qquad \uparrow x_i \quad \downarrow U(x_i)$$

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(Images courtesy xkcd.org)

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- Many applications in symmetric cryptography:
(efficient) encryption, identification, authentication, ...

(Images courtesy xkcd.org)

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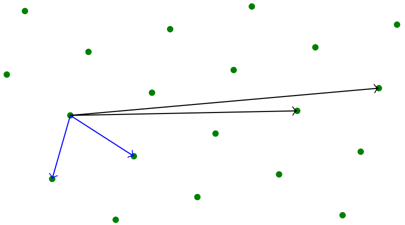
   - ✗ Large circuits that need much preprocessing
   - ✗ No "post-quantum" construction under standard assumptions

Why Not Try Lattices?



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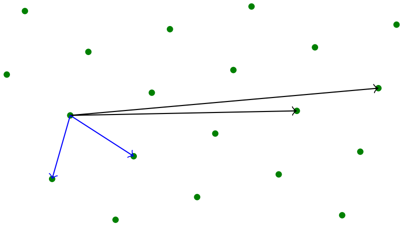

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Advantages of Lattice Crypto Schemes

- Simple & efficient: linear, highly parallel operations
- Resist quantum attacks (so far)
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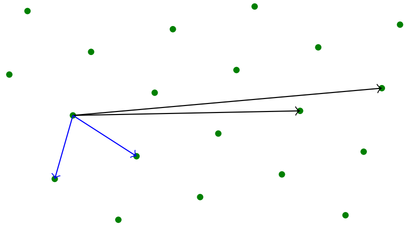

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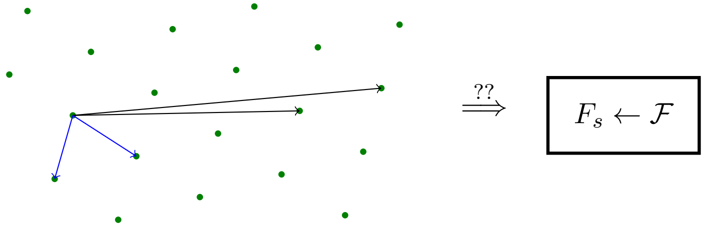

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✗✗ We don't even have practical PRGs from lattices: biased errors

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   Also gives more practical PRGs, GGM-type PRFs, encryption, . . .

Synthesizers and PRFs [NaorReingold'95]

Synthesizer

- A deterministic function $S\colon D \times D \to D$ s.t. for any $m = \text{poly}$:
  for $a_1, \ldots, a_m,\ b_1, \ldots, b_m \leftarrow D$,

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 & b_1 & b_2 & \cdots \\
\hline\hline
a_1 & S(a_1, b_1) & S(a_1, b_2) & \cdots \\
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- Alternative view: an (almost) length-squaring PRG with locality: maps $D^{2m} \to D^{m^2}$, and each output depends on only 2 inputs.

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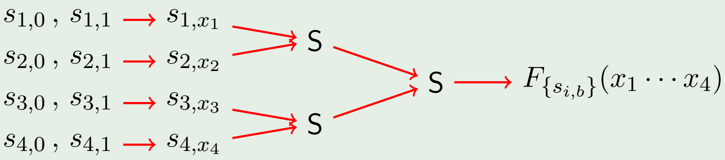

- Security: the queries $F_\ell(x_\ell)$ and $F_r(x_r)$ define (pseudo)random inputs $a_1, a_2, \ldots \in D$ and $b_1, b_2, \ldots \in D$ to synthesizer S.

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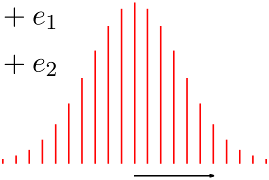

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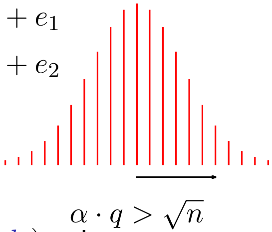

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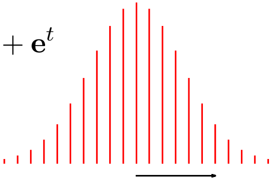

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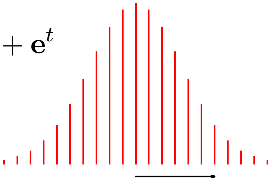

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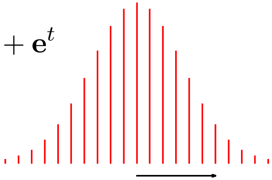

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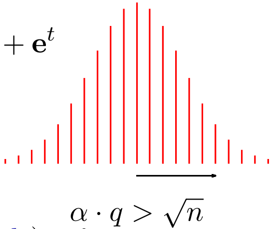

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  - ⋆ There's an $\exp((\alpha q)^2)$-time attack! [AG'11]

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   Random $\mathbf{t}$'s (with fresh samples) $\Rightarrow$ random self-reduction.

   Lets us amplify success probabilities (both search & decision):

   non-negl on uniform $\mathbf{s} \leftarrow \mathbb{Z}_q^n \quad \Longrightarrow \quad \approx 1$ on <u>any</u> $\mathbf{s} \in \mathbb{Z}_q^n$

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$$\begin{aligned} \mathbf{a} \ , \ b' &= b + \langle \mathbf{t}, \mathbf{a} \rangle \\ &= \langle \mathbf{s} + \mathbf{t}, \mathbf{a} \rangle + e. \end{aligned}$$

   Random $\mathbf{t}$'s (with fresh samples) $\Rightarrow$ random self-reduction.

   Lets us amplify success probabilities (both search & decision):

   non-negl on uniform $\mathbf{s} \leftarrow \mathbb{Z}_q^n \quad \Longrightarrow \quad \approx 1$ on <u>any</u> $\mathbf{s} \in \mathbb{Z}_q^n$

3. Multiple secrets: $(\mathbf{a}, b_1 \approx \langle \mathbf{s}_1, \mathbf{a} \rangle, \ldots, b_t \approx \langle \mathbf{s}_t, \mathbf{a} \rangle)$ vs. $(\mathbf{a}, b_1, \ldots, b_t)$.

   Simple hybrid argument, since $\mathbf{a}$'s are *public*.

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- Suppose $\mathcal{D}$ solves decision-LWE: it 'perfectly' distinguishes between pairs $(\mathbf{a}, b = \langle \mathbf{s}, \mathbf{a} \rangle + e)$ and $(\mathbf{a}, b)$.

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- Don't really need $\boxed{\text{prime } q = \text{poly}(n)}$ [P'09,ACPS'09,MM'11,MP'12]

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Learning With Errors (LWE) [Regev'05]

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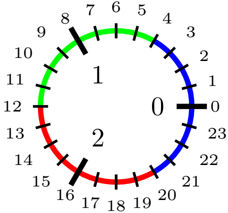
✗ What about $\mathbf{E}_{i,j}$?

Synthesizer must be deterministic...

“Learning With Rounding” (LWR) [Banerjee Peikert Rosen’12]

- IDEA: generate errors deterministically by rounding $\mathbb{Z}_q$ to a “sparse” subset (e.g. $\mathbb{Z}_p$).

  (Common in decryption to remove error.)



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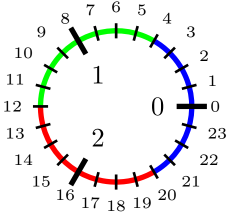

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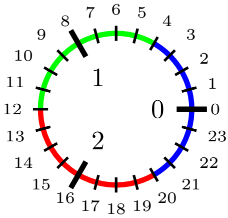

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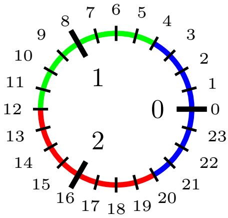

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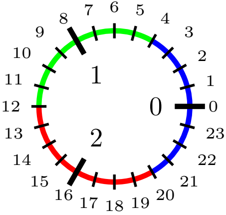

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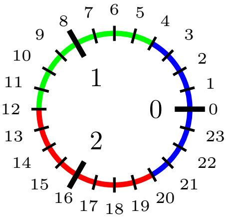

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  Proof idea: w.h.p., $(\,\mathbf{a}\,,\ \lfloor \langle \mathbf{a}, \mathbf{s} \rangle + e \rceil_p\,) = (\,\mathbf{a}\,,\ \lfloor \langle \mathbf{a}, \mathbf{s} \rangle \rceil_p\,)$

  and $(\,\mathbf{a}\,,\ \lfloor \text{Unif}(\mathbb{Z}_q) \rceil_p\,) = (\,\mathbf{a}\,,\ \text{Unif}(\mathbb{Z}_p)\,)$

Properties of LWR

1. Random Self Reducubility:
   On input $A, R(As)$, output $AX, R(As)$, for random $X \in \mathbb{Z}_q^{n \times n}$,

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- Synthesizer $S\colon \mathbb{Z}_q^{n\times n} \times \mathbb{Z}_q^{n\times n} \to \mathbb{Z}_p^{n\times n}$ is $\quad S(\mathbf{A}, \mathbf{S}) = \lfloor \mathbf{A} \cdot \mathbf{S} \rceil_p$.

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- "Tower" of public moduli $q_d > q_{d-1} > \cdots > q_0$.
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$$\left\lfloor \left\lfloor \left\lfloor \mathbf{S}_{1,x_1} \cdot \mathbf{S}_{2,x_2}^t \right\rceil_q \cdot \left\lfloor \mathbf{S}_{3,x_3} \cdot \mathbf{S}_{4,x_4}^t \right\rceil_q \right\rceil_q \cdot \left\lfloor \left\lfloor \mathbf{S}_{5,x_5} \cdot \mathbf{S}_{6,x_6}^t \right\rceil_q \cdot \left\lfloor \mathbf{S}_{7,x_7} \cdot \mathbf{S}_{8,x_8}^t \right\rceil_q \right\rceil_q \right\rceil_q$$

More Efficient?

Ring Learning With Errors (RLWE) [LPR'10]

- For (e.g.) n a power of 2, define "cyclotomic" polynomial rings

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- Shorter description/faster computation (using FFT/NTT).

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Has small(ish) TC^0 circuit, via CRT and reduction to subset-sum.

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- Repeat for $s_2, s_3, \ldots$ until $F''''''(x) = \lfloor a_x \rceil_p =$ Uniform func. $\square$

Open Questions 1

1. Better (worst-case) hardness for LWR, e.g. for $q/p = \sqrt{n}$?

   (The proof from LWE relies on approx factor and modulus $= n^{\omega(1)}$.)

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   [BCGR'13]:

   - LWR $\leq$ LWE for $\lceil q/p \rceil = n^{O(1)}$ (uses ideas from [FGKP'06]).
   - Adaptations of [AG'11] and [BKL'03] to LWR.

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