Robustness in Geometric Computation

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Mysore Park Workshop, August, 2013

Outline

1. Nonrobustness
   - In Computational Geometry: Example
   - The Exact Geometric Computation (EGC) Approach
   - Open Problems & Further Directions

2. Real Root Isolation
   - Two Algorithms & Their Analysis
   - Open Problems & Further Directions

What is Non-robustness?

Behaviour of a program

Inconsistent results, Infinite loops, “Crashes” (Segmentation Fault).

Implications

- Disasters caused by malfunctioning of software (e.g., Ariane crash in 1996).
- Reduces programmer’s effectiveness – time spent in debugging programs.

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- Efficient algorithms for Convex Hulls, Voronoi Diagrams, Delaunay Triangulations ...

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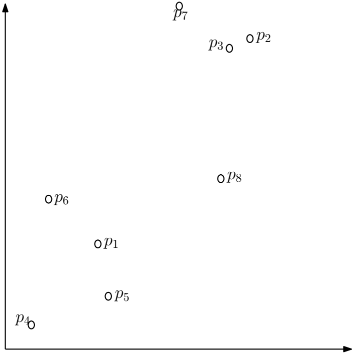
> *"Robustness problems are often a cause of frustration when implementing geometric algorithms."*

But haven't the Numerical Analysts addressed nonrobustness?

Backward stability, Forward-error analysis.

Computing Convex Hull – An Incremental Algorithm



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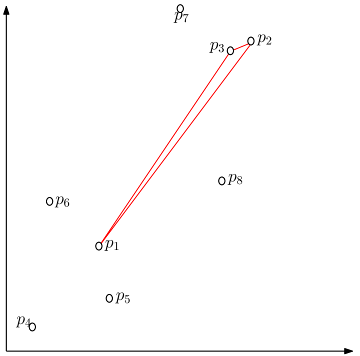

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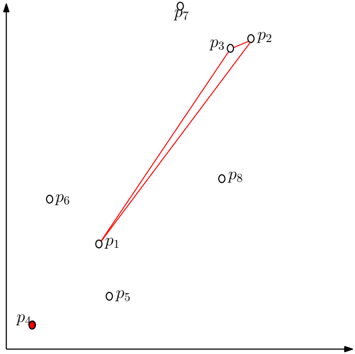

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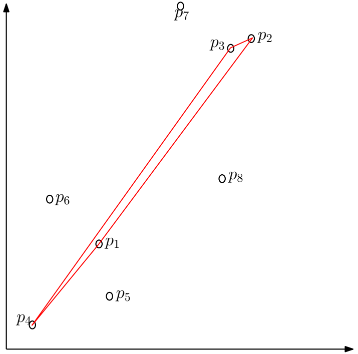

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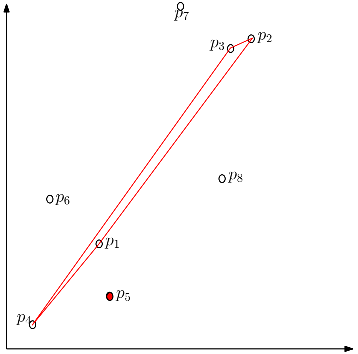

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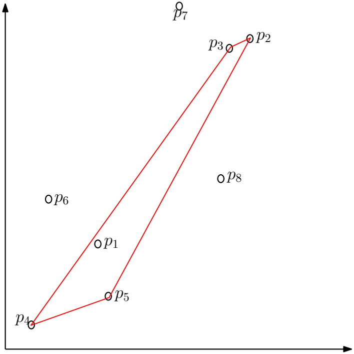

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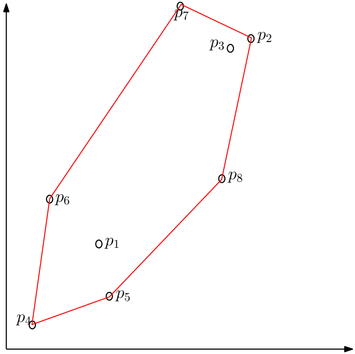

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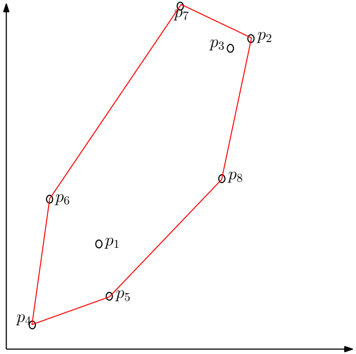

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$\text{orient}(p, q, r) = L$ iff (p, q, r) is left turn.

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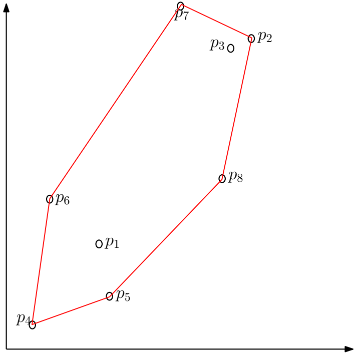

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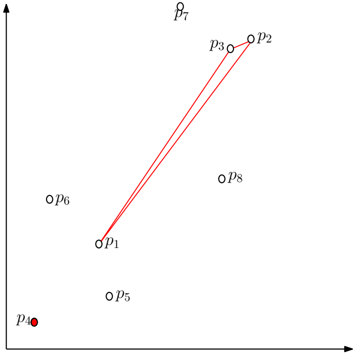
Given $CH(p_1, \ldots, p_k)$ and p. If $\forall i$, $\text{orient}(p_i, p_{i+1}, p) = L$ then p is in CH.



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Algebraic form of orient

Sign of
$((q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x))$.
"L" is $+1$, "R" is -1, "S" is 0.
In practice, fl_orient(p, q, r).

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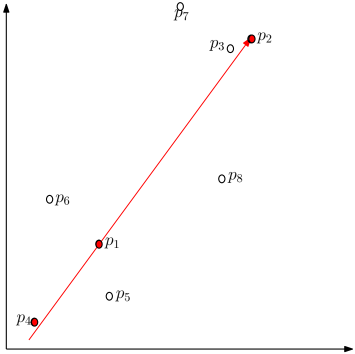

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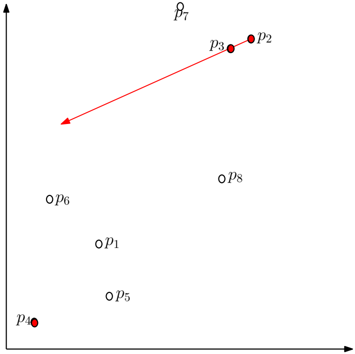

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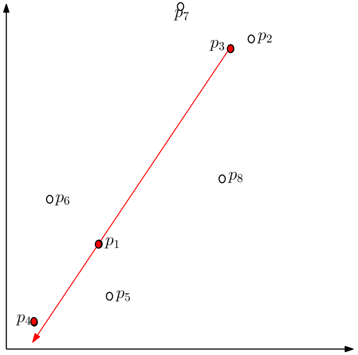

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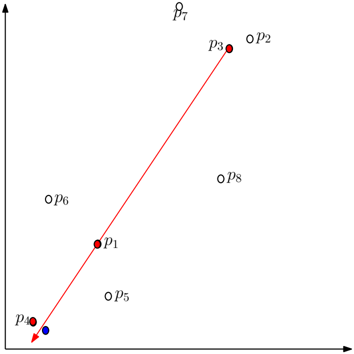

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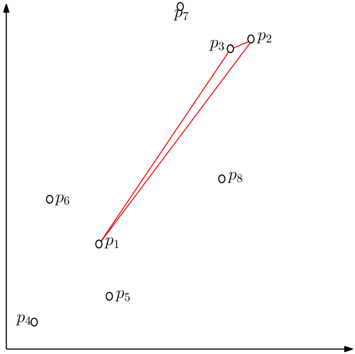

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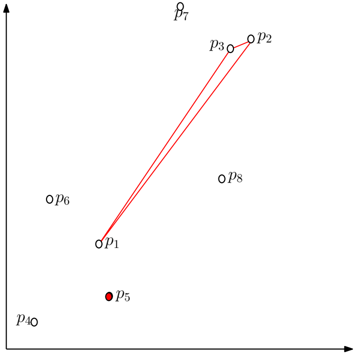

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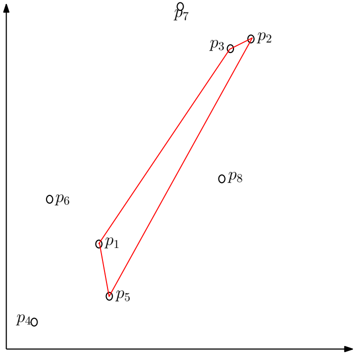

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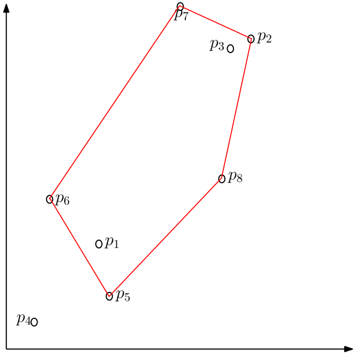

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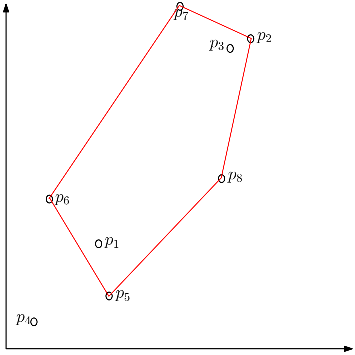

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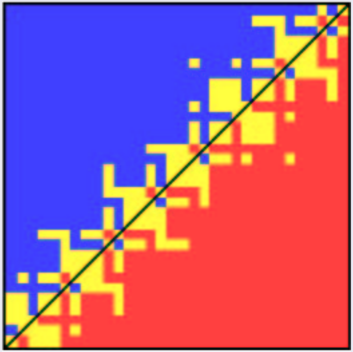
p_1, p_2, p_3, p_4 are almost collinear.

Geometry of fl_orient



- $p = (0.5, 0.5), q = (12, 12), r = (24, 24)$.
- $\text{fl\_orient}((p_x + iu, p_y + ju), q, r)$, $0 \leq i, j \leq 255$.
- $u = 2^{-53}$.

- Left turn
- Right turn
- Collinear

Kettner et al.'06

Classroom Examples of Robustness Problems in Geom. Comp.

The Exact Geometric Computation (EGC) Approach

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Geometric Algorithms [Yap'94]

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- Numerical representation of the geometric objects.
  E.g., floating-point representation of coordinates.
- Characterize the combinatorial structure by verifying the discrete relations using numerical computations.
  E.g., using the orientation predicate.

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- Numerical representation of the geometric objects.
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- Characterize the combinatorial structure by verifying the discrete relations using numerical computations.
  E.g., using the orientation predicate.

Cause of Non-robustness

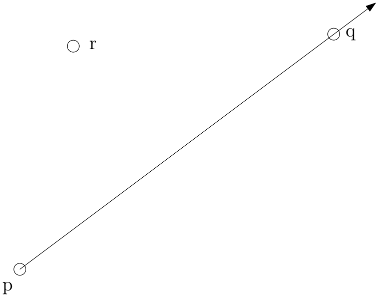
Numerical errors may give incorrect characterization.

The Exact Geometric Computation (EGC) Approach

The EGC solution

- Compute correct discrete relations between geometric objects.
- Correct sign evaluation of geometric predicates.
- What are geometric predicates?

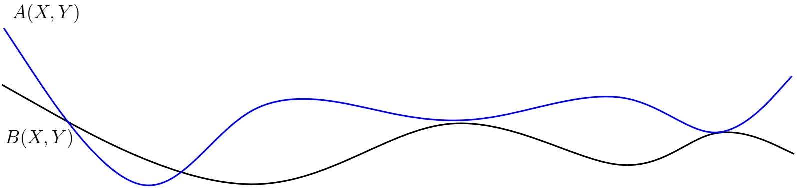
The Exact Geometric Computation (EGC) Approach



Orientation predicate

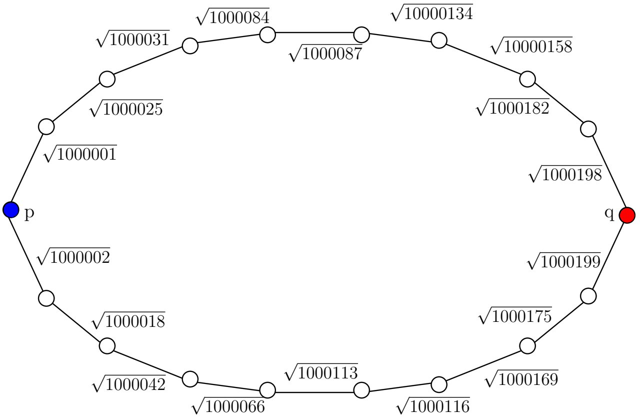
Whether r is to the left, right, or collinear with (p, q)?

The Exact Geometric Computation (EGC) Approach



Whether two algebraic curves intersect?

The Exact Geometric Computation (EGC) Approach



Which is the shorter path?

Almost equal – difference is smaller than 10^{-36}.

The Exact Geometric Computation (EGC) Approach

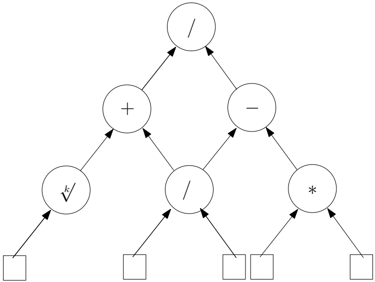
Geometric Predicates

Sign of (multivariate) polynomials evaluated at real algebraic numbers

Real Algebraic Numbers

Real roots of integer polynomials in one variable.
E.g. $\pm\sqrt{n}$, for positive integer n, $\frac{1+\sqrt{5}}{2}$, but not π, e.

The Exact Geometric Computation (EGC) Approach



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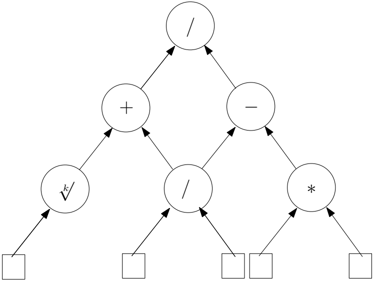

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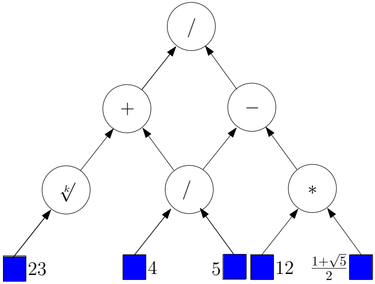

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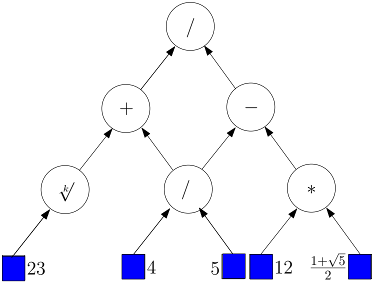

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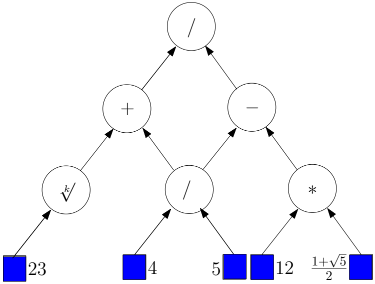

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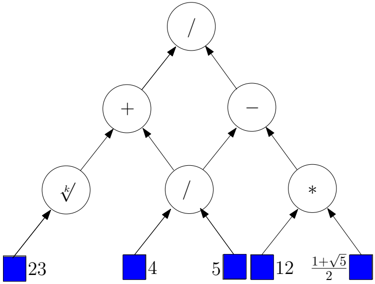

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- Input $G(\alpha)$ and $G(\beta)$.

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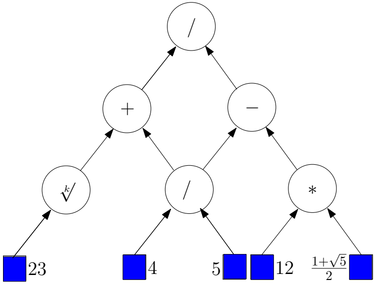

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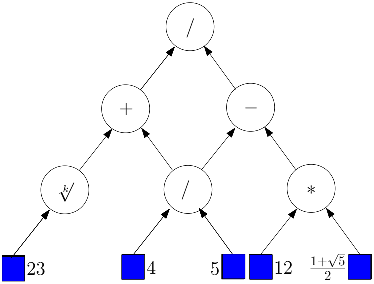

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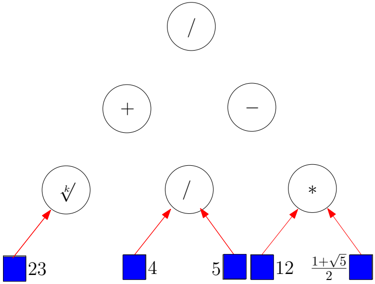

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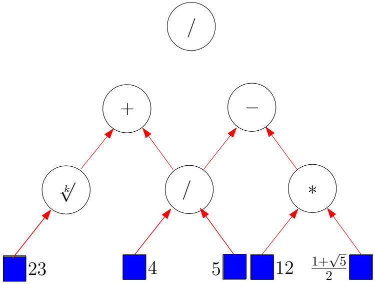

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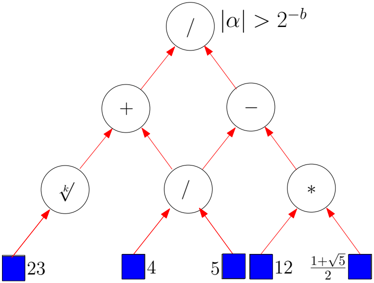

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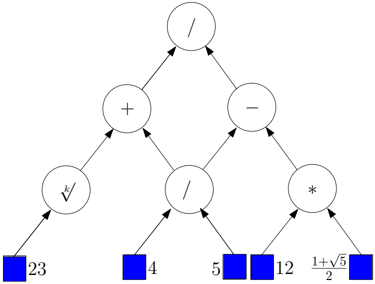

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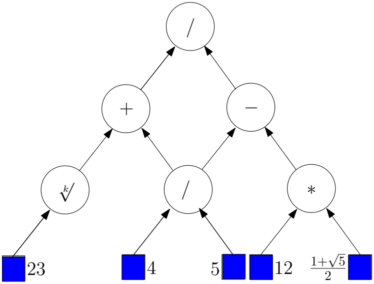

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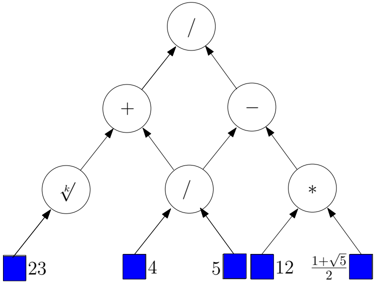

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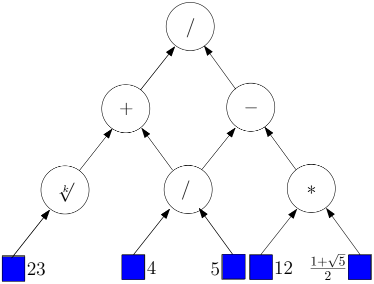

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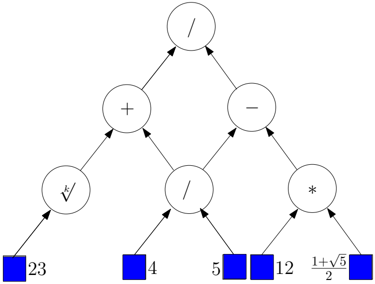

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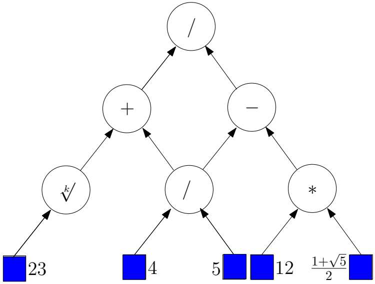

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The Exact Geometric Computation Approach

Implementations

- Leda reals – `http://www.mpi-inf.mpg.de/LEDA/`
- Core Library – `http://www.cs.nyu.edu/exact/core`

Used in CGAL – `www.cgal.org`

Theoretical Foundations of EGC

Ideal World

- Algorithms in Real RAM model.
- Computes a function $f : \mathbb{R} \to \mathbb{R}$.

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The EGC World, [Yap, 2003]

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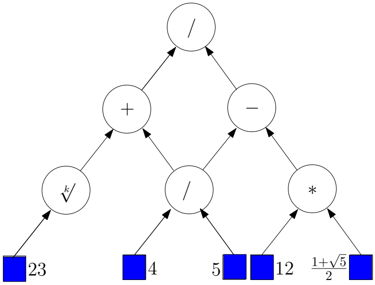

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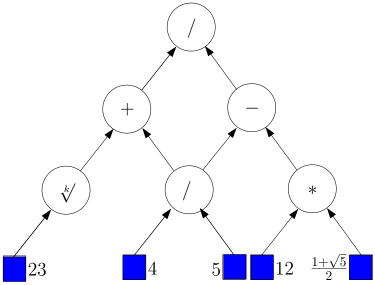

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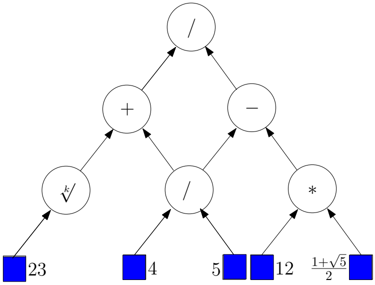

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Zero Problem for exp, log Expressions, Richardson'07

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Schanuel's Conjecture

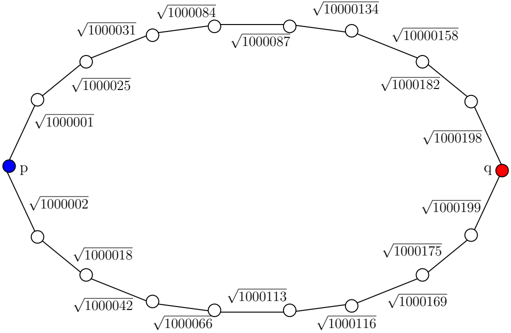
Given n complex numbers $z_1, \ldots, z_n$ linearly independent over $\mathbb{Q}$.
At least n transcendental numbers in $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$.

Generalizes Lindemann-Weierstrass Theorem

$z_1, \ldots, z_n$ are n linearly independent algebraic numbers.

Complexity of Algebraic Numbers

Which algebraic numbers can be relatively approximated in poly time?



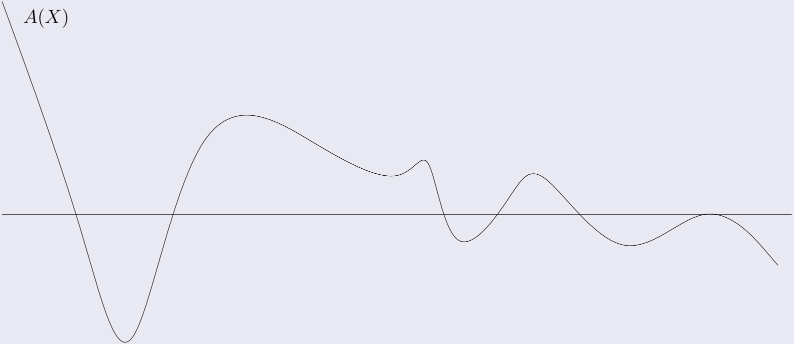
Sum of Square Roots

- $\sum_{i=1}^{k} \sqrt{a_i} - \sum_{i=1}^{k} \sqrt{b_i}$, $|a_i|, |b_i| \leq N$.
- $S(N,k)$, the minimum positive absolute value of the sum.
- Current bounds: $S(N,k) \gtrapprox N^{-2^k}$.
- Hope: $S(N,k) \gtrapprox N^{-k}$.

Real Root Isolation

The Problem

Given $A(X) \in \mathbb{Z}[X]$, degree d.



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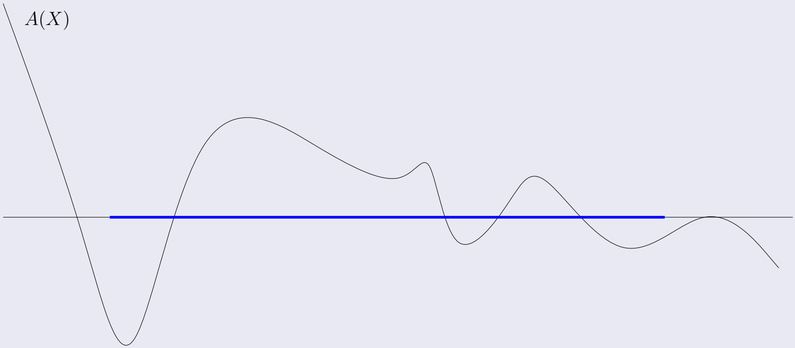

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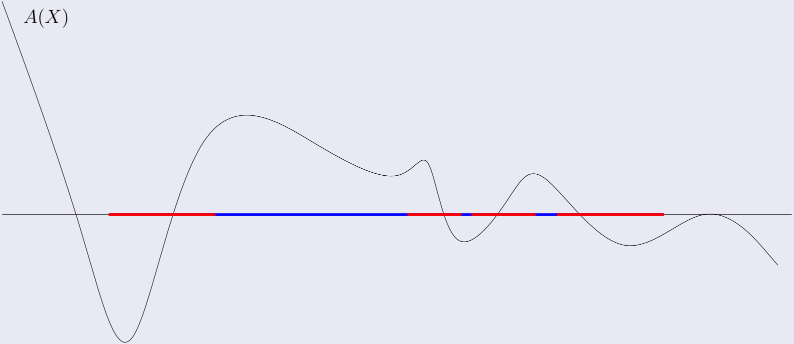

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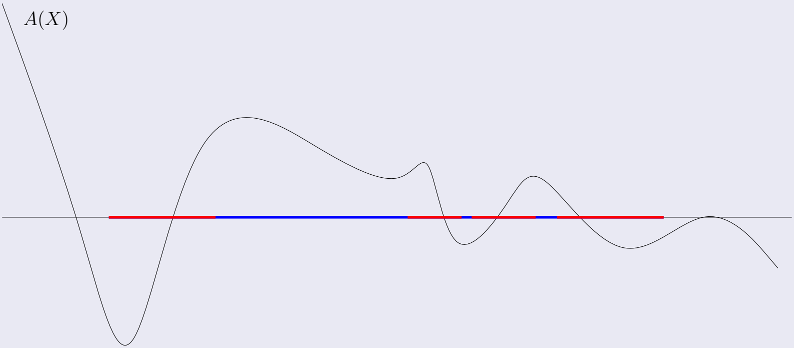

Assumption

All the roots are of multiplicity one, i.e. $\texttt{GCD}(A(X), A'(X)) = 1$.

Selective History

Classical Work

- Descartes, Newton, Fourier, Sturm, Vincent,
- Weierstrass, Obreshkoff, Ostrowski, Weyl, Henrici, ...

Modern Work – Complexity and Implementation

- Schönhage, Smale, Pan (optimal complexity results), ...
- Collins, Johnson, Krandick, Bini, Mehlhorn, Sagraloff, ...

Relevance

Fundamental problem in computational algebra, used in Cylindrical Algebraic Decomposition, in Ray Tracing, Computer Aided Design, for verifying conjectures, ...

A General Subdivision Algorithm

Input: $A(X) \in \mathbb{Z}[X]$ of degree d, and I_0.
Output: Isolating intervals for roots of $A(X)$ in I_0.

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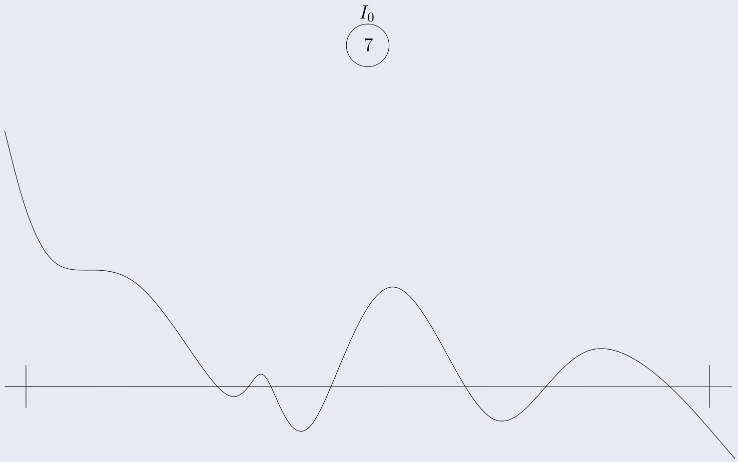
RootIsol($A(X), I$)

1. If $\text{Count}(A, I) = 0$ then return.
2. If $\text{Count}(A, I) = 1$ then output I and return.
3. Let m be the midpoint of $I = (I_l, I_r)$.
4. Recurse on $(A(X), (I_l, m))$ and $(A(X), (m, I_r))$.

Illustration



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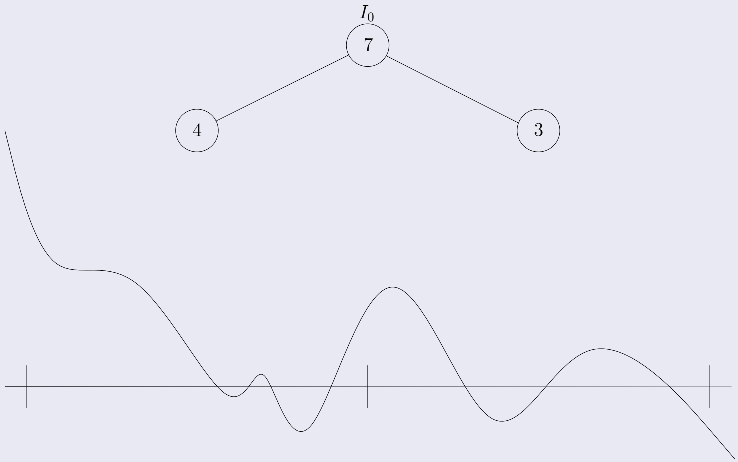

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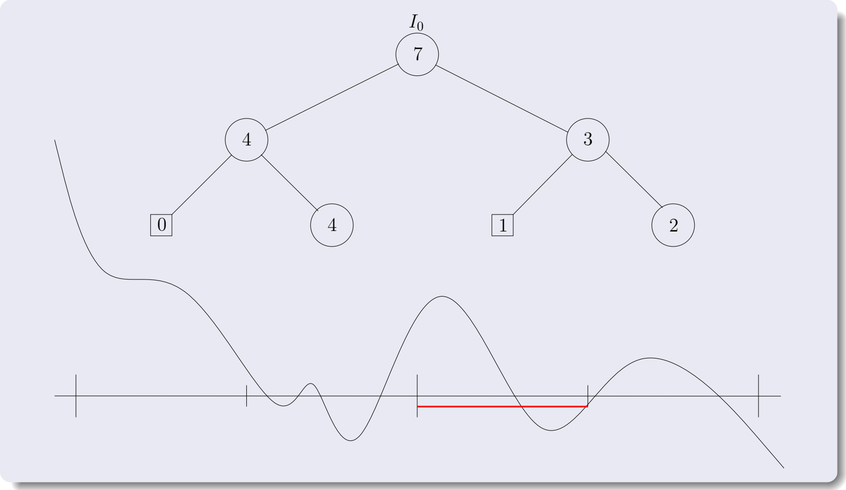

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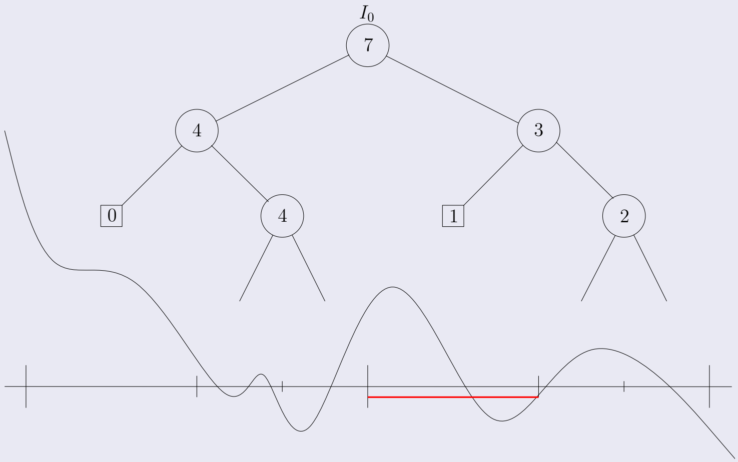

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The Sequence

- $\overline{A} := (A_0 = A, A_1 = A', \ldots, A_k)$, $A_{i+1} := \quad \texttt{rem}(A_{i-1}, A_i)$.

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- $\overline{A} := (A_0 = A, A_1 = A', \ldots, A_k)$, $A_{i+1} := -\mathtt{rem}(A_{i-1}, A_i)$.
- $A(x) = (x^2 - x - 1)$, $A'(x) = 2x - 1$
- $x^2 - x - 1 = (2x-1)\frac{(2x-1)}{4} - \frac{5}{4} = \frac{4x^2-4x+1}{4} - \frac{5}{4}$.
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- Given $c \in \mathbb{R}$, evaluate $\overline{A}$ at c, i.e., $(A_0(c), A_1(c), \ldots, A_k(c))$.
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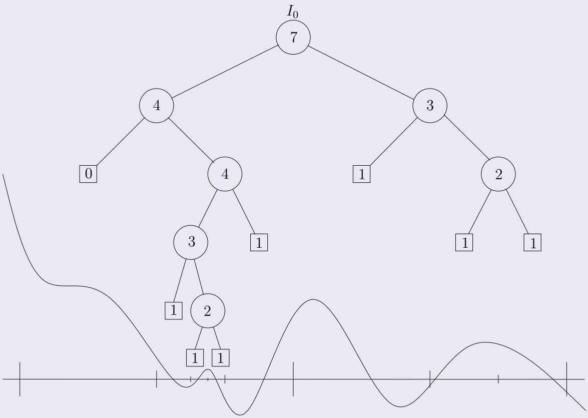
Sturm's Theorem, 1829

No. of real roots of $A(x)$ in (c, d) = $\mathrm{Var}(\overline{A}; c) - \mathrm{Var}(\overline{A}; d)$.

Complexity Analysis – Sturm's Algorithm, Davenport'85



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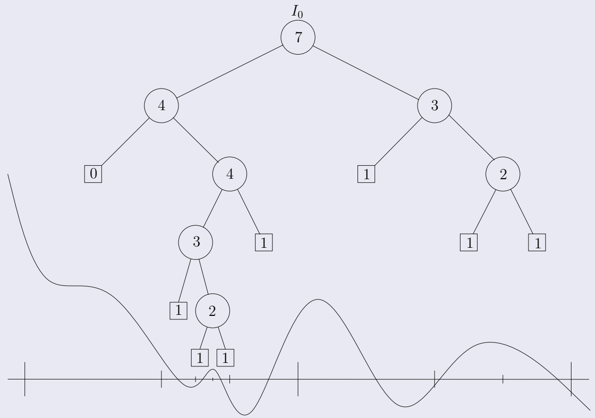

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Complexity Analysis – Sturm's Algorithm

Measure of Complexity

Root Separation of A, $\mathrm{sep}(A) := \min\{|\alpha - \beta| : \alpha, \beta \in Z(A) \subseteq \mathbb{C}\}$.

Complexity Analysis – Sturm's Algorithm

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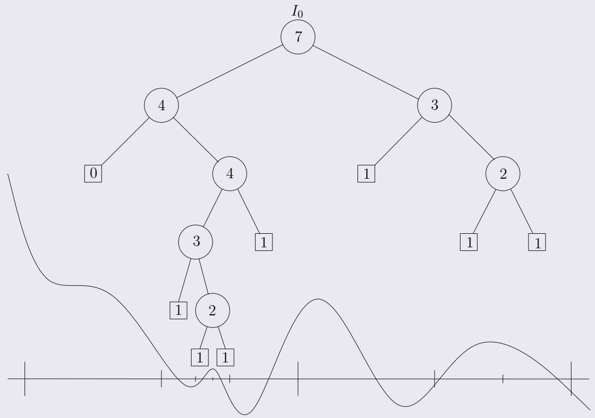
Bounds

- $A(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{Z}[x]$, degree d, $|a_i| \le 2^L$, $i = 0, \ldots, d$.
- $-\log \mathrm{sep}(A) = O(dL + d \log d)$.
- $w(I_0) < 2^L$.

Complexity Analysis – Sturm's Algorithm



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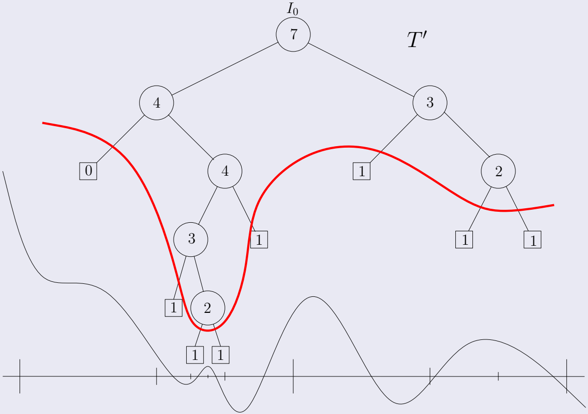

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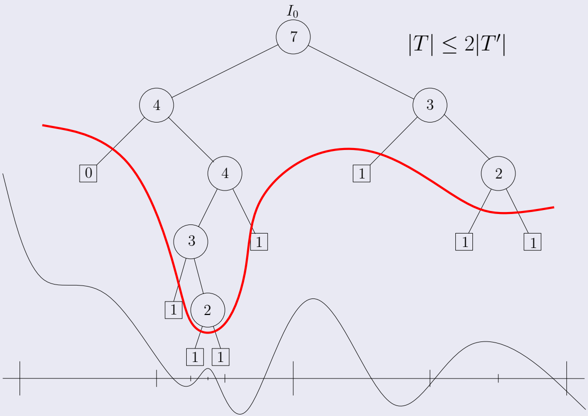

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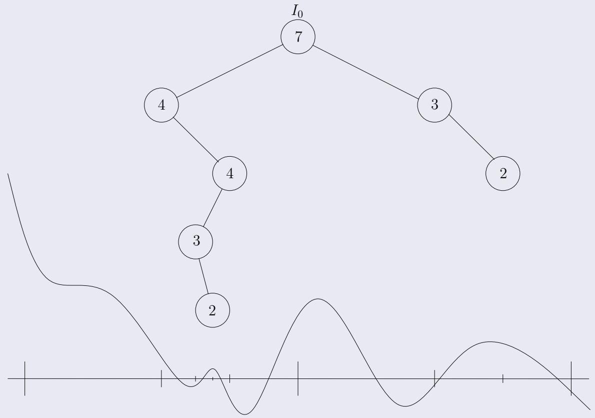

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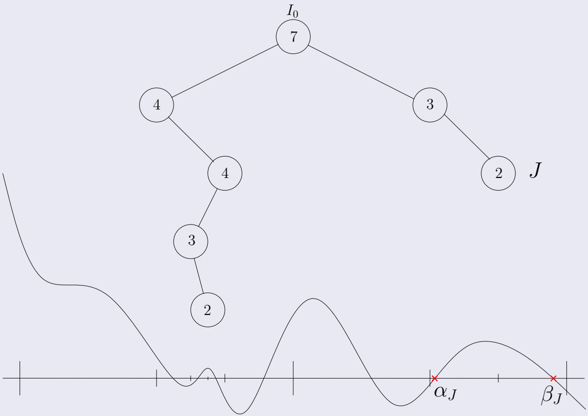

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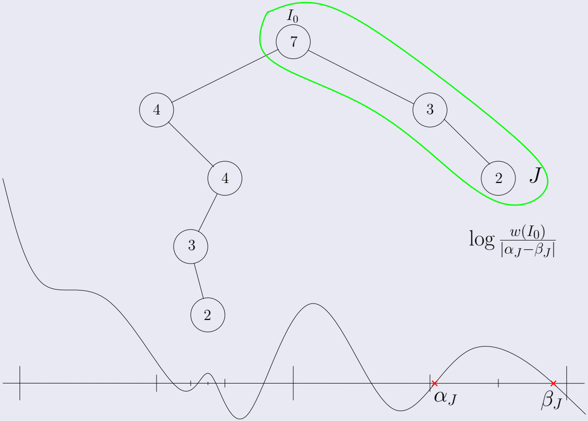

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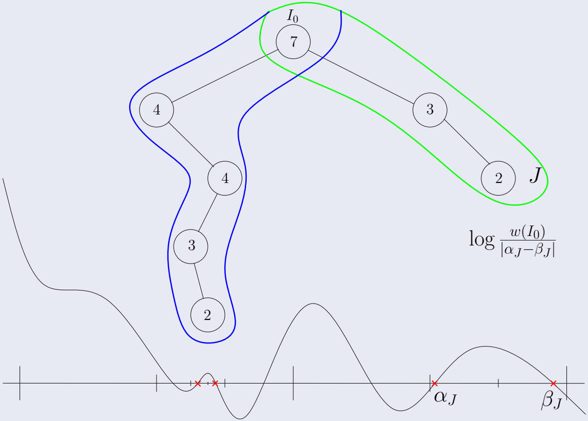

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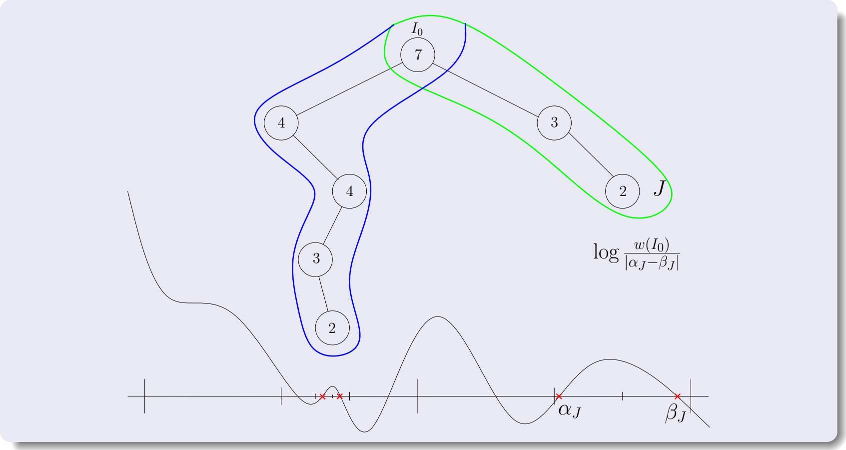

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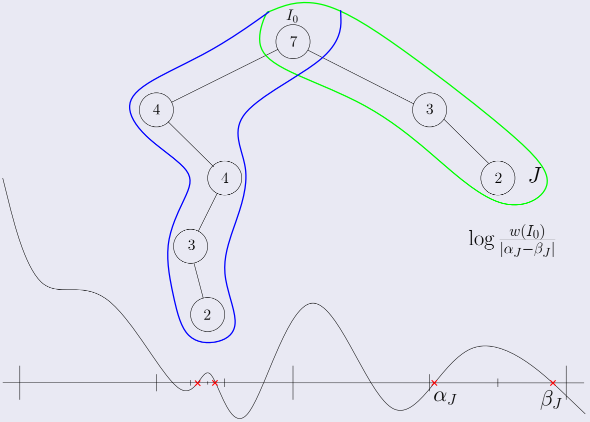
Complexity Analysis – Sturm's Algorithm



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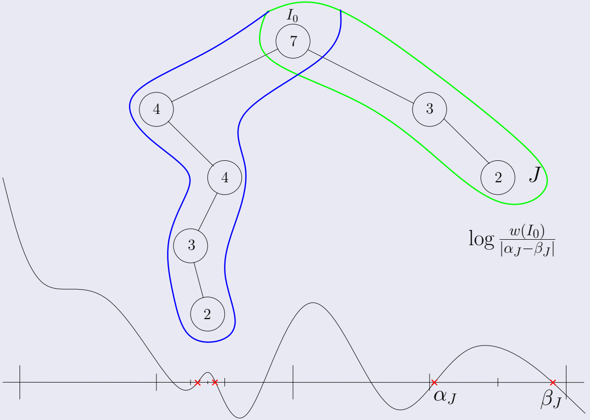
Complexity Analysis – Sturm's Algorithm



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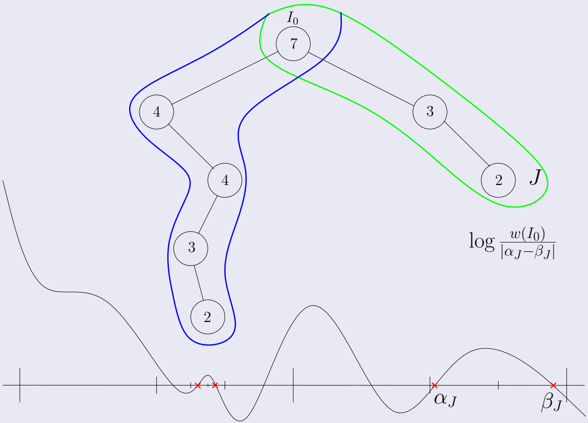
Complexity Analysis – Sturm's Algorithm



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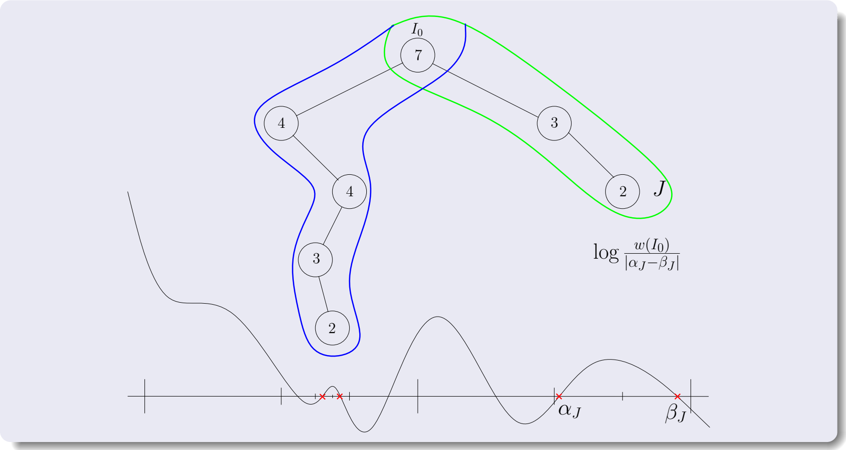

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Some Remarks on Sturm's Algorithm

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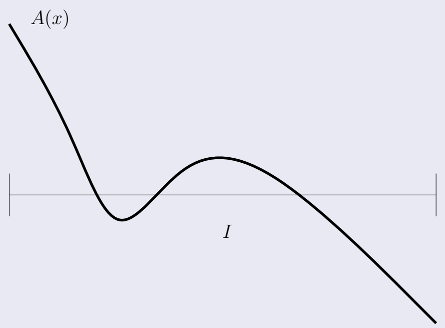
- It's not the first algorithm that comes to mind.

An Idea For Real Root Isolation



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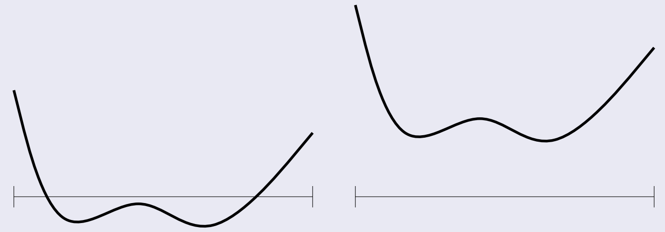

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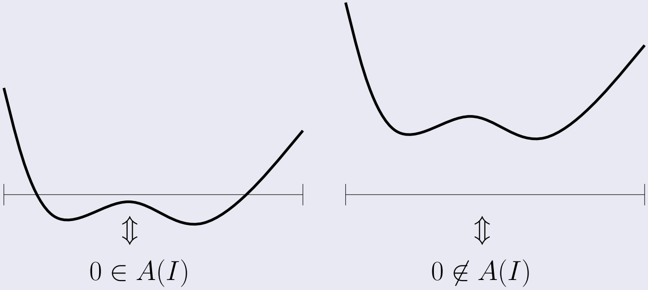

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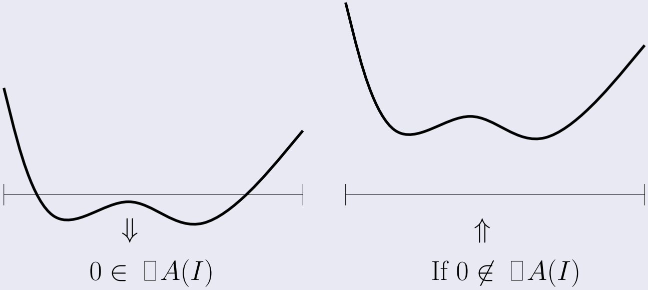

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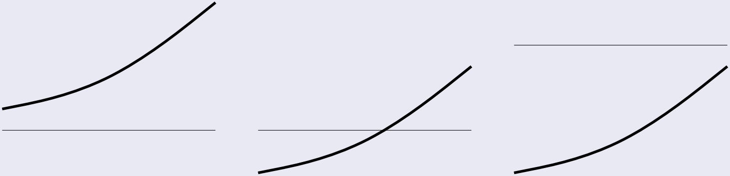

$\Uparrow$

If $0 \notin \square A'(I)$

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The Algorithm, Mitchell'90

Input: $A(X) \in \mathbb{R}[X]$ of degree d, and I_0.
Output: Isolating intervals for roots of $A(X)$ in I_0.

EVAL Algorithm

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2. While Q is not empty do
3. Remove an interval I from Q.
4. If $0 \notin \square A(I)$ or $0 \notin \square A'(I)$ then stop.
5. Else
   Subdivide I into two halves and push them on Q.

Assumption

$A(x)$ is square-free, no multiple roots.

Implementing the Box-function

Let $A(x) = \sum_{i=0}^{n} a_i x^i$, $a_i \in \mathbb{R}$.

Interval Arithmetic

- $[a,b] + [c,d] := [a+c, b+d]$.
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Two Properties of Box-functions

- Conservative: $A(I) \subseteq \square A(I)$.
- Convergent: $I_1 \supset I_2 \supset I_3 \supset \cdots \supset \{x\}$ then $\square A(I_j) \to A(x)$.

EVAL: Bounds on Recursion Tree Size

Goal – Real Root Isolation

$A \in \mathbb{Z}[x]$ square-free, degree d, with L-bit coefficients.

$$\text{Aim: } O(d(L + \log d))$$

Similar bounds for Sturm's method.

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Result

$O(d(L + r))$, where r is the number of real roots in input interval.

The EVAL Algorithm

Input: $A(X) \in \mathbb{R}[X]$ of degree d, and I_0.
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EVAL Algorithm: EVAL(A, I_0)

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Definition

$P(I_0)$ – partition of I_0 at the leaves of the subdivision tree EVAL(A, I_0).

EVAL: An Integral Bound on Tree Size

Stopping Function, Burr-Krahmer-Yap'09

- $G : \mathbb{R} \to \mathbb{R}_{\geq 0}$.
- If $\exists x \in I$ such that $w(I)G(x) \leq 1$ then I is terminal.
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- $2 \int_{I_0} G(x)dx = \sum_{I \in P(I_0)} 2 \int_I G(x)dx \geq \sum_{I \in P(I_0)} 1 = \#P(I_0)$.

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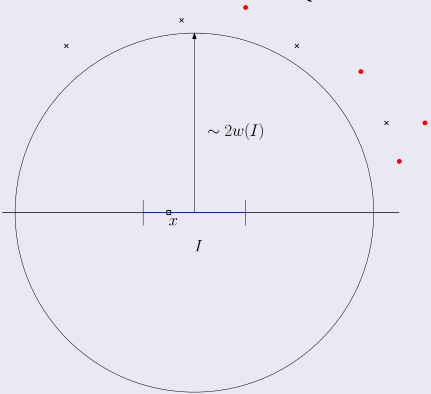
Intuitively, we expect $G(x) \sim 2\min\left\{\frac{1}{|x-Z(A)|}, \frac{1}{|x-Z(A')|}\right\}$.



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The Stopping Function (Burr-Krahmer, 2012)

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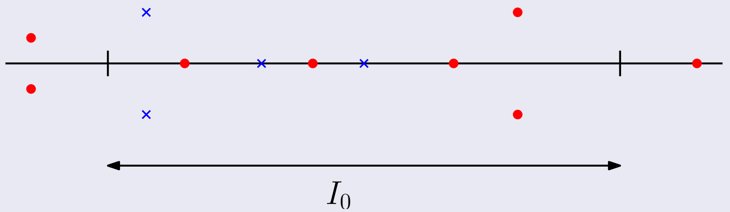
Key Property

- $\left|\frac{A'(x)}{A(x)}\right| \leq S(x), \left|\frac{A''(x)}{A(x)}\right| \leq S^2(x), \ldots, \left|\frac{A^{(k)}(x)}{A(x)}\right| \leq S^k(x), \ldots.$
- $\sum_{k>0}\left|\frac{A^{(k)}(x)}{k!A(x)}\right|\left(\frac{w(I)}{2}\right)^k \leq \sum_{k>0}\frac{1}{k!}\left(\frac{S(x)w(I)}{2}\right)^k < 1.$

EVAL: Bounding the Integral, S.-Yap'12

$$S(x) := \sum_{\alpha \in Z(A)} \frac{1}{|x-\alpha|} \qquad T(x) := \sum_{\alpha' \in Z(A')} \frac{1}{|x-\alpha'|}.$$

$$\#P(I_0) \leq \int_{I_0} \min\{S(x), T(x)\}\, dx$$



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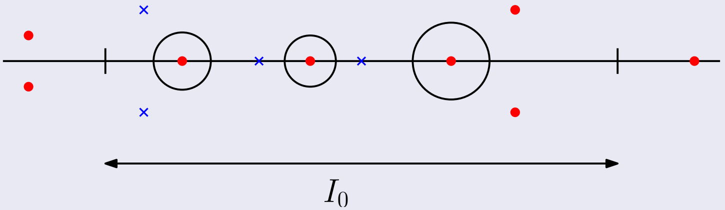

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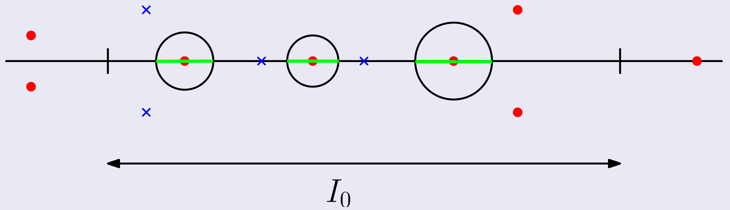

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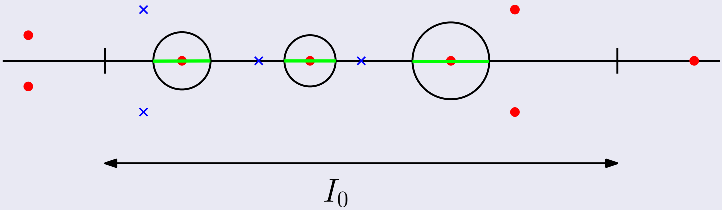

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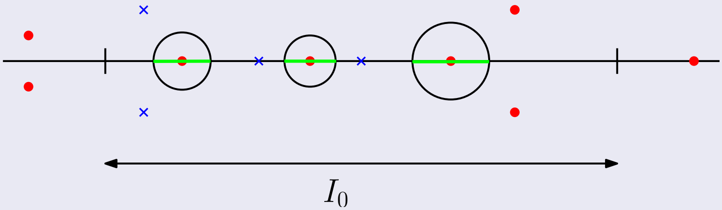

$$\int_{I_1} T(x)dx = O(dr) \text{ and } \int_{I_0 \setminus I_1} S(x)dx = O(d(L + \log d)).$$

Real Root Isolation – Beyond Polynomials

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Root Clustering of Holomorphic Functions, Sagraloff-S.-Yap'13

- Pellet's Theorem: Disc $D(m,r) \subseteq \mathbb{C}$ and $|f_k(m)r^k| > \sum_{j \neq k} |f_j(m)| r^j$ then $D(m,r)$ contains k roots.
- Darboux's theorem.
- Soft-predicates: Given $\varepsilon \geq 0$, if $x \leq \varepsilon$ then treat x as 0.

The Subdivision Paradigm

In Other Contexts

- Root Isolation of Zero Dimensional Systems [Moore (1966), Kearfott (1987), Stahl (1995)].
- Isotopic meshing of curves and surfaces [Snyder (1992), Plantinga-Vegter (2004), Lin-Yap (2011)].
- Marching cube algorithms [Lorensen-Cline (1987)].
- Robot Motion Planning [Brooks and Lozano-Perez (1983), Zhu-Latombe (1991)].
- Voronoi Diagrams of Polytopes, Polyhedron. [Vleugel-Overmars (1995), Li-S.-Yap (2012)].

Thank You!