Interlacing Families I: Bipartite Ramanujan Graphs of all Degrees

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Expander Graphs

Sparse regular well-connected graphs with many properties of random graphs.

Every set of vertices has many neighbors.
Random walks mix quickly.
Pseudo-random generators.
Error-correcting codes.
Used throughout Computer Science.
Spectral Expanders

Let $G$ be a graph and $A$ be its adjacency matrix

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}
$$

Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$
Spectral Expanders

Let $G$ be a graph and $A$ be its adjacency matrix

\[ \lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \]

If $d$-regular, then $A\mathbf{1} = d\mathbf{1}$ so $\lambda_1 = d$.

If bipartite then eigs are symmetric about zero so $\lambda_n = -d$.
Spectral Expanders

**Definition:** $G$ is a good expander if all non-trivial eigenvalues are small.
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- $K_d$ and $K_{d,d}$ have all nontrivial eigs 0.
Definition: $G$ is a good expander if all non-trivial eigenvalues are small.

Alon-Boppana'86: For every $\epsilon > 0$, every sufficiently large $d$-regular graph has a nontrivial eigenvalue greater than $2\sqrt{d - 1} - \epsilon$.

Challenge: construct infinite families.
Ramanujan Graphs: \(2\sqrt{d} - 1\)

**Definition:** \(G\) is Ramanujan if all non-trivial eigs have absolute value at most \(2\sqrt{d} - 1\).
**Ramanujan Graphs:** \(2\sqrt{d} - 1\)

**Definition:** \(G\) is **Ramanujan** if all non-trivial eigs have absolute value at most \(2\sqrt{d} - 1\)

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**Margulis, Lubotzky-Phillips-Sarnak’88:** Infinite sequences of Ramanujan graphs exist for \(d = p + 1\)

**Friedman’08:** A random \(d\)-regular graph is almost Ramanujan: \(2\sqrt{d} - 1 + \epsilon\)
Main Result

Theorem. Infinite families of bipartite Ramanujan graphs exist for every $d \geq 3$. 
Main Result

**Theorem.** Infinite families of bipartite Ramanujan graphs exist for every $d \geq 3$.

Proof is elementary, doesn’t use number theory. Not explicit.

Based on a new existence argument: method of interlacing families of polynomials.
Bilu-Linial’06 Approach

Find an operation which doubles the size of a graph without blowing up its eigenvalues.

\[ \begin{bmatrix} -d & -2\sqrt{d - 1} & 0 & 2\sqrt{d - 1} & d \end{bmatrix} \]
Bilu-Linial’06 Approach

Find an operation which doubles the size of a graph without blowing up its eigenvalues.
Bilu-Linial’06 Approach

Find an operation which doubles the size of a graph without blowing up its eigenvalues.

\[
\begin{bmatrix}
    -d & -2\sqrt{d-1} & 0 & 2\sqrt{d-1} \\
\end{bmatrix}
\]
2-lifts of graphs
2-lifts of graphs

duplicate every vertex
2-lifts of graphs

duplicate every vertex
2-lifts of graphs

for every pair of edges:
leave on either side (parallel),
or make both cross
2-lifts of graphs

for every pair of edges: leave on either side (parallel), or make both cross
2-lifts of graphs

for every pair of edges: leave on either side (parallel), or make both cross

$2^m$ possibilities
2-lifts of graphs

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\(n\) eigenvalues \(\{\lambda_1 \ldots \lambda_n\}\)
# 2-lifts of graphs

\[
\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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2-lifts of graphs

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0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
\]

2n eigenvalues \( \{\lambda_1 \ldots \lambda_n\} \cup \{\lambda'_1 \ldots \lambda'_n\} \)
Eigenvalues of 2-lifts (Bilu-Linial)

Given a 2-lift of $G$, create a signed adjacency matrix $A_s$ with a -1 for crossing edges and a 1 for parallel edges.

$$
\begin{pmatrix}
0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}
$$
Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
The eigenvalues of the 2-lift are the union of the eigenvalues of \( A \) (old) and the eigenvalues of \( A_s \) (new)

\[
\{ \lambda'_1 \ldots \lambda'_n \} = eigs(A_s)
\]

\[
A_s = \begin{pmatrix}
0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}
\]
Eigenvalues of 2-lifts (Bilu-Linial)

**Theorem:**
The eigenvalues of the 2-lift are the union of the eigenvalues of $A$ (old) and the eigenvalues of $A_s$ (new)

**Conjecture:**
Every $d$-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d - 1}$
Eigenvalues of 2-lifts (Bilu-Linial)

**Theorem:**
The eigenvalues of the 2-lift are the union of the eigenvalues of $A$ (old) and the eigenvalues of $A_s$ (new)

**Conjecture:**
Every $d$-regular adjacency matrix $A$ has a signing $A_s$ with $||A_s|| \leq 2\sqrt{d - 1}$
Eigenvalues of 2-lifts (Bilu-Linial)

Conjecture:
Every d-regular adjacency matrix $A$ has a signing $A_s$ with $||A_s|| \leq 2\sqrt{d - 1}$

We prove this in the bipartite case.
Eigenvalues of 2-lifts (Bilu-Linial)

**Theorem:**
Every $d$-regular adjacency matrix $A$ has a signing $A_s$ with $\lambda_1(A_s) \leq 2\sqrt{d - 1}$
Eigenvalues of 2-lifts (Bilu-Linial)

**Theorem:**
Every d-regular **bipartite** adjacency matrix $A$ has a signing $A_S$ with $\|A_S\| \leq 2\sqrt{d - 1}$

**Trick:** eigenvalues of bipartite graphs are symmetric about 0, so only need to bound largest
Random Signings

**Idea 1:** Choose $s \in \{-1,1\}^m$ randomly.
Random Signings

Idea 1: Choose $s \in \{-1,1\}^m$ randomly.

Unfortunately,

$$\mathbb{E}\|A_s\| \gg 2\sqrt{d - 1}$$

(Bilu-Linial showed $O(\sqrt{d \log^3 d})$ when $A$ is nearly Ramanujan)
Random Signings

Idea 2: Observe that \( \lambda_1(A_s) = \lambda_{\text{max}}(\chi A_s) \)

where \( \chi A_s(x) := \det(xI - A_s) \)
Random Signings

Idea 2: Observe that \( \lambda_1(A_s) = \lambda_{\max}(\chi A_s) \)

where \( \chi A_s(x) := \det(xI - A_s) \)

Consider \( \mathbb{E}_{s \in \{\pm 1\}^m} \chi A_s(x) \)
Random Signings

Idea 2: Observe that $\lambda_1(A_s) = \lambda_{\text{max}}(\chi A_s)$ where $\chi A_s(x) := \det(xI - A_s)$

Consider $\mathbb{E}_{s\in\{\pm1\}^m} \chi A_s(x)$

Usually useless, but not here!

$\{\chi A_s\}_{s\in\{\pm1\}^m}$ is an interlacing family.

$\exists S$ such that $\lambda_{\text{max}}(\chi A_s) \leq \lambda_{\text{max}}(\mathbb{E}\chi A_s)$
3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$.

$$\exists S \text{ such that } \lambda_{\text{max}}(\chi A_s) \leq \lambda_{\text{max}}(\mathbb{E}\chi A_s)$$
3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$.

   $\exists S \text{ such that } \lambda_{\text{max}}(\chi_{A_S}) \leq \lambda_{\text{max}}(\mathbb{E}\chi_{A_S})$

2. Calculate the expected polynomial.

   $\mathbb{E}\chi_{A_S}(x) = \mu_G(x)$
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3. Bound the largest root of the expected poly.

$$\lambda_{max}(\mu_G(x)) \leq 2\sqrt{d - 1}$$
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1. Show that some poly does as well as the $E$.

$$\exists S \text{ such that } \lambda_{max}(\chi A_s) \leq \lambda_{max}(E\chi A_s)$$

2. Calculate the expected polynomial.

$$E\chi A_s(x) = \mu_G(x)$$

3. Bound the largest root of the expected poly.

$$\lambda_{max}(\mu_G(x)) \leq 2\sqrt{d-1}$$
Step 2: The expected polynomial

**Theorem** [Godsil-Gutman’81]

For any graph $G$,

$$\mathbb{E}_s \left[ \chi_{A_s}(x) \right] = \mu_G(x)$$

the matching polynomial of $G$
The matching polynomial
(Heilmann-Lieb ‘72)

\[ \mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i \]

\[ m_i = \text{the number of matchings with } i \text{ edges} \]
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]

one matching with 0 edges
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]

7 matchings with 1 edge
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]
Proof that $\mathbb{E}_s \left[ \chi A_s(x) \right] = \mu_G(x)$

Expand $\mathbb{E}_s \left[ \det(xI - A_s) \right]$ using permutations

$$
\begin{array}{ccccccc}
  x & \pm 1 & 0 & 0 & \pm 1 & \pm 1 \\
\pm 1 & x & \pm 1 & 0 & 0 & 0 \\
0 & \pm 1 & x & \pm 1 & 0 & 0 \\
0 & 0 & \pm 1 & x & \pm 1 & 0 \\
\pm 1 & 0 & 0 & \pm 1 & x & \pm 1 \\
\pm 1 & 0 & 0 & 0 & \pm 1 & x \\
\end{array}
$$
Proof that $\mathbb{E}_s \left[ \chi A_s(x) \right] = \mu_G(x)$

Expand $\mathbb{E}_s \left[ \det(xI - A_s) \right]$ using permutations

*same edge:*

\[
\begin{array}{ccccccc}
x & \pm1 & 0 & 0 & \pm1 & \pm1 \\
\pm1 & x & \pm1 & 0 & 0 & 0 \\
0 & \pm1 & x & \pm1 & 0 & 0 \\
0 & 0 & \pm1 & x & \pm1 & 0 \\
\pm1 & 0 & 0 & \pm1 & x & \pm1 \\
\pm1 & 0 & 0 & 0 & \pm1 & x \\
\end{array}
\]

*same value*
Proof that $\mathbb{E}_s \left[ \chi A_s(x) \right] = \mu_G(x)$

Expand $\mathbb{E}_s \left[ \det(xI - A_s) \right]$ using permutations

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>±1</th>
<th>0</th>
<th>0</th>
<th>±1</th>
<th>±1</th>
</tr>
</thead>
<tbody>
<tr>
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<td>x</td>
<td>±1</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>±1</td>
<td>x</td>
</tr>
</tbody>
</table>

same edge: ±1
same value
Proof that $\mathbb{E}_s [ \chi A_s (x) ] = \mu_G (x)$

Expand $\mathbb{E}_s [ \det (xI - A_s) ]$ using permutations

Get 0 if hit any 0s
Proof that $\mathbb{E}_s[\chi A_s(x)] = \mu_G(x)$

Expand $\mathbb{E}_s[\det(xI - A_s)]$ using permutations

Get 0 if take just one entry for any edge
Proof that \( \mathbb{E}_s \left[ \chi A_s(x) \right] = \mu_G(x) \)

Expand \( \mathbb{E}_s \left[ \det(xI - A_s) \right] \) using permutations

Only permutations that count are involutions
Proof that \[ \mathbb{E}_s \left[ \chi_{A_s}(x) \right] = \mu_G(x) \]

Expand \[ \mathbb{E}_s \left[ \det(xI - A_s) \right] \] using permutations

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Correspond to matchings
Proof that \( \mathbb{E}_s \left[ \chi_{A_s}(x) \right] = \mu_G(x) \)

Expand \( \mathbb{E}_s \left[ \det(xI - A_s) \right] \) using permutations

\[
\begin{pmatrix}
  x & \pm 1 & 0 & 0 & \pm 1 & \pm 1 \\
  \pm 1 & x & \pm 1 & 0 & 0 & 0 \\
  0 & \pm 1 & x & \pm 1 & 0 & 0 \\
  0 & 0 & \pm 1 & x & \pm 1 & 0 \\
  \pm 1 & 0 & 0 & \pm 1 & x & \pm 1 \\
  \pm 1 & 0 & 0 & 0 & \pm 1 & x \\
\end{pmatrix}
\]

Only permutations that count are involutions

Correspond to matchings
3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$.

$$\exists S \text{ such that } \lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})$$

2. Calculate the expected polynomial.

$$\mathbb{E}\chi_{A_s}(x) = \mu_G(x) \quad \text{[Godsil-Gutman’81]}$$

3. Bound the largest root of the expected poly.

$$\lambda_{max}(\mu_G(x)) \leq 2\sqrt{d-1}$$
3-Step Proof Strategy

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The matching polynomial
(Heilmann-Lieb ‘72)

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**Theorem** (Heilmann-Lieb)
al all the roots are real
The matching polynomial
(Heilmann-Lieb ‘72)

\[ \mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i \]

**Theorem** (Heilmann-Lieb)
all the roots are real
and have absolute value at most \(2\sqrt{d - 1}\)

**Proof**: simple, based on recurrences.
3-Step Proof Strategy

1. Show that some poly does as well as the expected polynomial.
   \[ \exists S \text{ such that } \lambda_{\text{max}}(\chi_{A_S}) \leq \lambda_{\text{max}}(\mathbb{E}\chi_{A_S}) \]

2. Calculate the expected polynomial.
   \[ \mathbb{E}\chi_{A_S}(x) = \mu_G(x) \quad \text{[Godsil-Gutman’81]} \]

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   \[ \lambda_{\text{max}}(\mu_G(x)) \leq 2\sqrt{d-1} \quad \text{[Heilmann-Lieb’72]} \]
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1. Show that some poly does as well as the $E$.
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$$\exists S \text{ such that } \lambda_{\text{max}}(\chi A_s) \leq \lambda_{\text{max}}(\mathbb{E}\chi A_s)$$

Implied by:

“$\{\chi A_s\}_{s \in \{\pm 1\}^m}$ is an interlacing family.”
Averaging Polynomials

Basic Question: Given $p_0, p_1$ when are the roots of the $p_i(x)$ related to roots of $\mathbb{E} p_i(x)$?
Averaging Polynomials

Basic Question: Given $p_0, p_1$ when are the roots of the $p_i(x)$ related to roots of $\mathbb{E}_i p_i(x)$?

Answer: Certainly not always

\[ p_0 = (x-1)^2 \]
\[ p_1 = (x+1)^2 \]

\[ \frac{1}{2} (p_0 + p_1) \]
No real roots!
Averaging Polynomials

Basic Question: Given $p_0, p_1$ when are the roots of the $p_i(x)$ related to roots of $\mathbb{E}_i p_i(x)$?

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No real roots!
Averaging Polynomials

Basic Question: Given $p_0, p_1$, when are the roots of the $p_i(x)$ related to roots of $\mathbb{E}_i p_i(x)$?

But sometimes it works:
A Sufficient Condition

Basic Question: Given $p_0, p_1$ when are the roots of the $p_i(x)$ related to roots of $\mathbb{E}_i p_i(x)$?

Answer: When they have a common interlacing.

Definition. $q = \prod_{i=1}^{n-1} (x - \alpha_i)$ interlaces $p = \prod_{i=1}^{n} (x - \beta_i)$ if

$$\beta_n \leq \alpha_{n-1} \leq \beta_{n-1} \ldots \leq \alpha_1 \leq \beta_1.$$
Theorem. If $p_0, p_1$ have a common interlacing, \[ \exists i \quad \lambda_{\text{max}}(p_i) \leq \lambda_{\text{max}}(E_i p_i) \]
Theorem. If \( p_0, p_1 \) have a common interlacing, \( \exists i \quad \lambda_{max}(p_i) \leq \lambda_{max}(E_i p_i) \)

Proof.
Theorem. If $p_0, p_1$ have a common interlacing, then $\exists i \quad \lambda_{max}(p_i) \leq \lambda_{max}(\sum_i p_i)$.

Proof.
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Proof.
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Proof.
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Proof.

So $\lambda_{max}(p_0) \leq \lambda_{max} (\mathbb{E} p_i)$
Theorem. If $p_0, p_1$ have a common interlacing, then $\exists i \quad \lambda_{\text{max}}(p_i) \leq \lambda_{\text{max}}(\mathbb{E}ip_i)$

Proof.

So $\lambda_{\text{max}}(p_0) \leq \lambda_{\text{max}}(\mathbb{E}p_i)$
**Interlacing Family of Polynomials**

**Definition:** \( \{p_s\}_{s \in \{0,1\}^m} \) is an interlacing family if can be placed on the leaves of a tree so that when every node is the sum of leaves below, sets of siblings have common interlacings.
Interlacing Family of Polynomials

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Interlacing Family of Polynomials

\[ \sum_{s \in \{0,1\}^m} p_s = p_\emptyset \]

\[ p_0 \]
\[ p_{00} \]
\[ p_{01} \]
\[ p_{10} \]
\[ p_{11} \]

**Theorem**: There is an \( s \) so that

\[ \lambda_{\text{max}}(p_s) \leq \lambda_{\text{max}}(p_\emptyset) \]
**Interlacing Family of Polynomials**

\[ \sum_{s \in \{0,1\}^m} p_s = p_\emptyset \]

**Theorem:** There is an \( s \) so that

\[ \lambda_{max}(p_s) \leq \lambda_{max}(p_\emptyset) \]

**Proof:** By common interlacing, one of \( p_0, p_1 \) has

\[ \lambda_{max} \leq \lambda_{max}(p_\emptyset) \]
Interlacing Family of Polynomials

\[ \sum_{s \in \{0,1\}^m} p_s = p_\emptyset \]

**Theorem:** There is an \( s \) so that

\[ \lambda_{max}(p_s) \leq \lambda_{max}(p_\emptyset) \]

**Proof:** By common interlacing, one of \( p_0, p_1 \) has \( \lambda_{max} \leq \lambda_{max}(p_\emptyset) \)
Interlacing Family of Polynomials

\[ \sum_{s \in \{0,1\}^m} p_s = p_\emptyset \]

**Theorem:** There is an \( s \) so that

\[ \lambda_{max}(p_s) \leq \lambda_{max}(p_\emptyset) \]

**Proof:** By common interlacing, one of \( p_{00}, p_{01} \) has

\[ \lambda_{max} \leq \lambda_{max}(p_0) \]
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An interlacing family

**Theorem:**

Let \( p_s(x) = \chi A_s(x) \)

\( \{p_s\}_{s \in \{\pm 1\}^m} \) is an interlacing family
An interlacing family

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**Lemma (easy):**

$p_0(x)$ and $p_1(x)$ have a common interlacing if and only if $\lambda p_0(x) + (1 - \lambda)p_1(x)$ is real rooted for all $0 \leq \lambda \leq 1$
To prove interlacing family

Let \( p_{s_1, \ldots, s_k}(x) = \mathbb{E}_{s_{k+1}, \ldots, s_m} \left[ p_{s_1, \ldots, s_m}(x) \right] \)

Leaves of tree = signings \( s_1, \ldots, s_m \)
Internal nodes = partial signings \( s_1, \ldots, s_k \)
To prove interlacing family

Let \( p_{s_1, \ldots, s_k}(x) = \mathbb{E}_{s_{k+1}, \ldots, s_m} [ p_{s_1, \ldots, s_m}(x) ] \)

Need to prove that for all \( s_1, \ldots, s_k \), \( \lambda \in [0, 1] \)

\[ \lambda p_{s_1, \ldots, s_k, 1}(x) + (1 - \lambda) p_{s_1, \ldots, s_k, -1}(x) \]

is real rooted
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\( s_1, \ldots, s_k \) are fixed
\( s_{k+1} \) is 1 with probability \( \lambda \) -1 with \( 1 - \lambda \)
\( s_{k+2}, \ldots, s_m \) are uniformly \( \pm 1 \)
Generalization of Heilmann-Lieb

Suffices to prove that

$$\mathbb{E}_{s \in \{\pm 1\}^m} [ p_s(x) ]$$ is real rooted

for every independent distribution on the entries of $s$
Generalization of Heilmann-Lieb

Suffices to prove that

\[ \mathbb{E}_{s \in \{\pm 1\}^m} \left[ p_s(x) \right] \]

is real rooted

for every independent distribution on the entries of \(s\):

\[ \sum_{s \in \{\pm 1\}^m} p_s(x) \prod_{i : s_i = 1} \lambda_i \prod_{i : s_i = -1} (1 - \lambda_i) \]

\[ \lambda_1, \ldots, \lambda_m \in [0, 1] \]
Transformation to PSD Matrices

Suffices to show real rootedness of

$$\mathbb{E}_{s \in \{\pm 1\}^m} p_s(x - d) = \mathbb{E}_{s \in \{\pm 1\}^m} \det(xI - (dI - A_s))$$
Transformation to PSD Matrices

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\[ \mathbb{E}_{s \in \{\pm 1\}^m} p_s(x - d) = \mathbb{E}_{s \in \{\pm 1\}^m} \det(xI - (dI - A_s)) \]

Why is this useful?

\[ A_s = \sum_{ij \in E} s_{ij} (\delta_i \delta_j^T + \delta_j \delta_i^T) \]
Transformation to PSD Matrices

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\[ A_s = \sum_{i,j \in E} s_{ij}(\delta_i \delta_j^T + \delta_j \delta_i^T) \]

\[ dI - A_s = \sum_{s_{ij} = 1} (\delta_i - \delta_j)(\delta_i - \delta_j)^T \]

\[ + \sum_{s_{ij} = -1} (\delta_i + \delta_j)(\delta_i + \delta_j)^T \]
Transformation to PSD Matrices

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\[ \mathbb{E}_s \det(xI - (dI - A_s)) = \mathbb{E} \det \left( xI - \sum_{ij \in E} v_{ij}v_{ij}^T \right) \]

where \[ v_{ij} = \begin{cases} 
(\delta_i - \delta_j) \text{ with probability } \lambda_{ij} \\
(\delta_i + \delta_j) \text{ with probability } (1-\lambda_{ij}) 
\end{cases} \]
Master Real-Rootedness Theorem

Given any independent random vectors $\nu_1, \ldots, \nu_m \in \mathbb{R}^d$, their expected characteristic polynomial

$$
\mathbb{E} \det \left( xI - \sum_i \nu_i \nu_i^T \right)
$$

has real roots.
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\]

has real roots.

How to prove this?
The Multivariate Method

A. Sokal, 90’s-2005:

“...it is often useful to consider the multivariate polynomial ... even if one is ultimately interested in a particular one-variable specialization”

Borcea-Branden 2007+: prove that univariate polynomials are real-rooted by showing that they are nice transformations of real-rooted multivariate polynomials.
Real Stable Polynomials

Definition: \( p \in \mathbb{R}[z_1, \ldots, z_n] \) is real stable if \( \text{imag}(z_i) > 0 \) for all \( i \) implies \( p(z_1, \ldots, z_n) \neq 0 \).
Real Stable Polynomials

Definition: \( p \in \mathbb{R}[z_1, \ldots, z_n] \) is real stable if \( \text{imag}(z_i) > 0 \) for all \( i \)
Implies \( p(z_1, \ldots, z_n) \neq 0 \).

no roots in the upper half-plane

univariate real stable = real-rooted
Excellent Closure Properties

**Definition:** \( p \in \mathbb{R}[z_1, \ldots, z_n] \) is *real stable* if \( \text{imag}(z_i) > 0 \) for all \( i \) implies \( p(z_1, \ldots, z_n) \neq 0 \).

If \( p \in \mathbb{R}[z_1, \ldots, z_n] \) is real stable, then so is

1. \( p(\alpha, z_2, \ldots, z_n) \) for any \( \alpha \in \mathbb{R} \)

2. \( (1 - \partial_{z_i})p(z_1, \ldots, z_n) \)
A Useful Real Stable Poly

Borcea-Brändén ‘08:
For PSD matrices $A_1, \ldots, A_k$

$$\det \left( \sum_i z_i A_i \right)$$

is real stable
A Useful Real Stable Poly

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$$\det(\sum_i z_i A_i)$$

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Plan: apply closure properties to this to show that $\mathbb{E} \det(xI - \sum_i v_i v_i^T)$ is real stable.
Central Identity

Suppose $\nu_1, \ldots, \nu_m$ are independent random vectors with $A_i := \mathbb{E}\nu_i\nu_i^T$. Then

$$\mathbb{E}\det \left( xI - \sum_{i} \nu_i \nu_i^T \right)$$

$$= \prod_{i=1}^{m} \left( 1 - \frac{\partial}{\partial z_i} \right) \det \left( xI + \sum_{i} z_i A_i \right) \bigg|_{z_1=\ldots=z_m=0}$$
Central Identity

Suppose \( \nu_1, \ldots, \nu_m \) are independent random vectors with \( A_i := \mathbb{E} \nu_i \nu_i^T \). Then

\[
\mathbb{E} \det \left( xI - \sum_i \nu_i \nu_i^T \right) = \prod_{i=1}^m \left( 1 - \frac{\partial}{\partial z_i} \right) \det \left( xI + \sum_i z_i A_i \right) \bigg|_{z_1=\cdots=z_m=0}
\]

Proof: easy, tomorrow.
Proof of Master Real-Rootedness Theorem

Suppose \( v_1, \ldots, v_m \) are independent random vectors with \( A_i := \mathbb{E}v_i v_i^T \). Then

\[
\mathbb{E} \det \left( xI - \sum_i v_i v_i^T \right) = \prod_{i=1}^m \left( 1 - \frac{\partial}{\partial z_i} \right) \det \left( xI + \sum_i z_i A_i \right) \bigg|_{z_1=\ldots=z_m=0}
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Proof of Master Real-Rootedness Theorem

Suppose $v_1, \ldots, v_m$ are independent random vectors with $A_i := \mathbb{E}v_i v_i^T$. Then

$$\mathbb{E} \det \left( xl - \sum_{i} v_i v_i^T \right) = \prod_{i=1}^{m} \left( 1 - \frac{\partial}{\partial z_i} \right) \det \left( xl + \sum_{i} z_i A_i \right) \bigg|_{z_1=\cdots=z_m=0}$$

Real Stable: closure under $(1-A)$
Proof of Master Real-Rootedness Theorem

Suppose $\nu_1, \ldots, \nu_m$ are independent random vectors with $A_i := \mathbb{E}\nu_i\nu_i^T$. Then

$$\mathbb{E}\det\left(xI - \sum_i \nu_i\nu_i^T\right) = \prod_{i=1}^m \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(xI + \sum_i z_i A_i\right) \bigg|_{z_1=\ldots=z_m=0}$$

Real stable: closure under $z_i = 0$.
Proof of Master Real-Rootedness Theorem

Suppose $v_1, \ldots, v_m$ are independent random vectors with $A_i := \mathbb{E}v_i v_i^T$. Then

$$
\mathbb{E} \det \left( xI - \sum_i v_i v_i^T \right) = \prod_{i=1}^m \left( 1 - \frac{\partial}{\partial z_i} \right) \det \left( xI + \sum_i z_i A_i \right) \bigg|_{z_1=\ldots=z_m=0}
$$

Real Stable $\Rightarrow$ Real Rooted!
The Whole Proof

$$\mathbb{E} \det(xI - \sum_i v_i v_i^T)$$ is real-rooted for all indep. $v_i$. 
The Whole Proof

\[ \mathbb{E} \chi_{A_s}(d - x) \text{ is real-rooted for all product distributions on signings.} \]

\[ \mathbb{E} \det(xI - \sum_i v_i v_i^T) \text{ is real-rooted for all indep. } v_i. \]
The Whole Proof

\[ \mathbb{E} \chi_{A_s}(x) \text{ is real-rooted for all product distributions on signings.} \]

\[ \mathbb{E} \text{det}(xI - \sum_i \nu_i \nu_i^T) \text{ is real-rooted for all indep. } \nu_i. \]
The Whole Proof

\[ \{ \chi_{A_s}(x) \}_{x \in \{\pm 1\}^m} \text{ is an interlacing family} \]

\[ \mathbb{E} \chi_{A_s}(x) \text{ is real-rooted for all product distributions on signings.} \]

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The Whole Proof

\[ \exists S \text{ such that } \lambda_{\text{max}}(\chi_{A_S}) \leq \lambda_{\text{max}}(\mathbb{E}\chi_{A_S}) \]

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\[ \mathbb{E}\det(xI - \sum_i \nu_i \nu_i^T) \text{ is real-rooted for all indep. } \nu_i. \]
3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$.
   \[ \exists S \text{ such that } \lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E} \chi_{A_s}) \]

2. Calculate the expected polynomial.
   \[ \mathbb{E} \chi_{A_s}(x) = \mu_G(x) \]

3. Bound the largest root of the expected poly.
   \[ \lambda_{max}(\mu_G(x)) \leq 2\sqrt{d-1} \]
Infinite Sequences of Bipartite Ramanujan Graphs

Find an operation which doubles the size of a graph without blowing up its eigenvalues.
Main Theme

Reduced the existence of a good matrix to:

1. Proving real-rootedness of an expected polynomial.

2. Bounding roots of the expected polynomial.
Main Theme

Reduced the existence of a good matrix to:

1. Proving real-rootedness of an expected polynomial.
   (general, using real stability)

2. Bounding roots of the expected polynomial.
   (specific, using combinatorics)
Tomorrow

Reduced the existence of a good matrix to:

1. Proving real-rootedness of an expected polynomial.
   (general, using real stability)

2. Bounding roots of the expected polynomial.
   (general, using new method)
Reduced the existence of a good matrix to:

1. Proving real-rootedness of an expected polynomial.
   (general, using real stability)

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   (general, using new method)

major implications in combinatorics, linear algebra + 1959 Kadison-Singer Conjecture.
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