Interlacing Families I: Bipartite Ramanujan Graphs of all Degrees

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Expander Graphs

Sparse regular well-connected graphs
with many properties of random graphs.

Every set of vertices has many neighbors.

Random walks mix quickly.

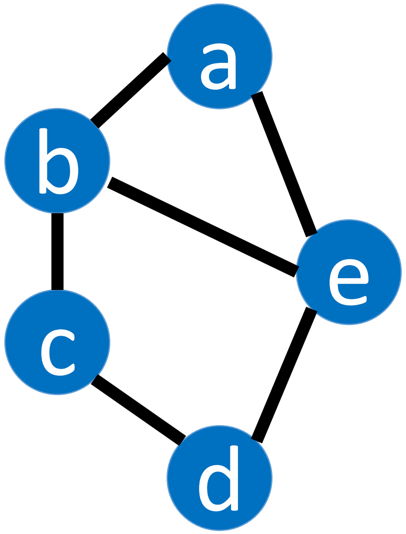
Pseudo-random generators.

Error-correcting codes.

Used throughout Computer Science.

Spectral Expanders

Let G be a graph and A be its adjacency matrix



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eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

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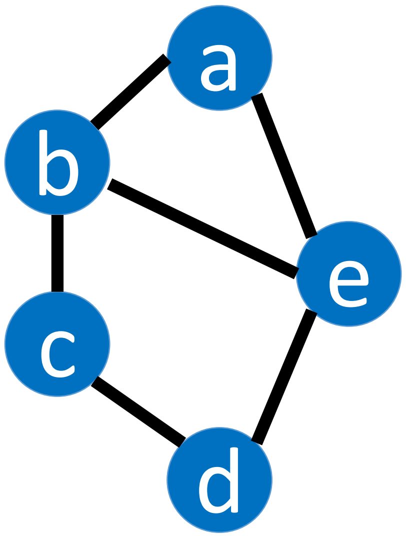

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eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

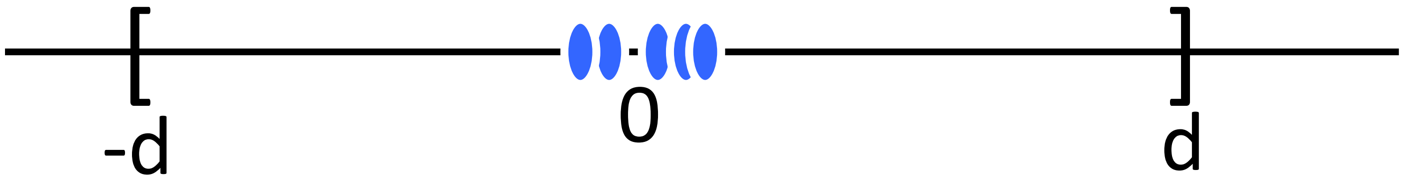
"trivial"

If d-regular, then $A\mathbf{1} = d\mathbf{1}$ so $\lambda_1 = d$

If bipartite then eigs are symmetric about zero so $\lambda_n = -d$

Spectral Expanders

Definition: *G* is a good expander
if all non-trivial eigenvalues are small



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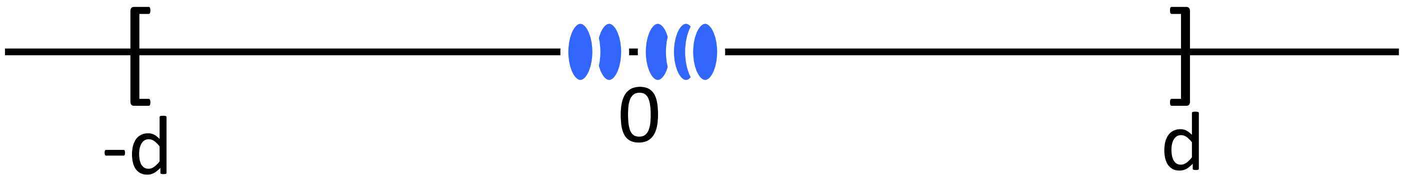

e.g. K_d and $K_{d,d}$ have all nontrivial eigs 0.

Spectral Expanders

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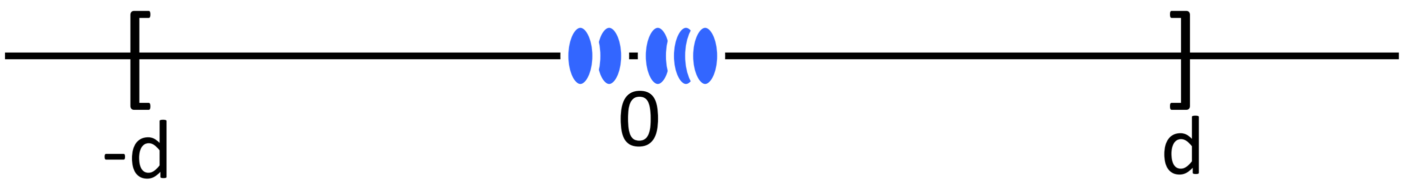
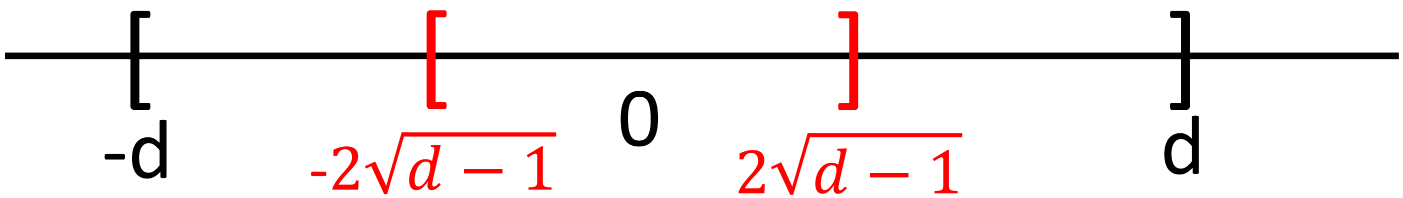


Challenge: construct infinite families.

Alon-Boppana'86: For every $\epsilon > 0$, every sufficiently large d-regular graph has a nontrivial eigenvalue greater than $2\sqrt{d-1} - \epsilon$

Ramanujan Graphs: $2\sqrt{d-1}$

Definition: G is **Ramanujan** if all non-trivial eigs have absolute value at most $2\sqrt{d-1}$



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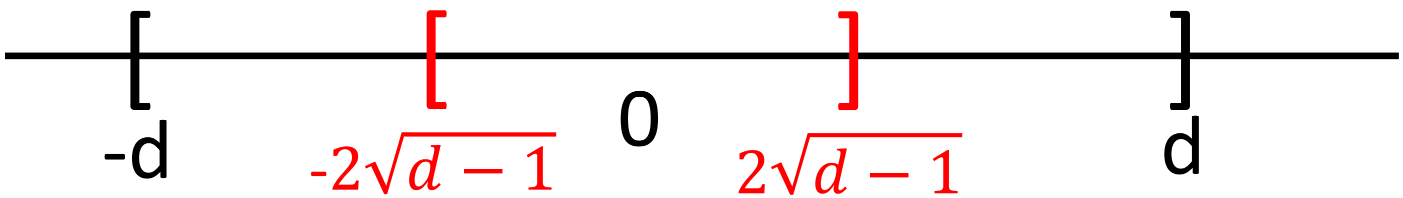

Margulis, Lubotzky-Phillips-Sarnak'88: Infinite sequences of Ramanujan graphs exist for $d = p + 1$

Friedman'08: A random d-regular graph is almost Ramanujan : $2\sqrt{d-1} + \epsilon$

Main Result

Theorem. Infinite families of bipartite Ramanujan graphs exist for every $d \geq 3$.

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Theorem. Infinite families of bipartite Ramanujan graphs exist for every $d \geq 3$.

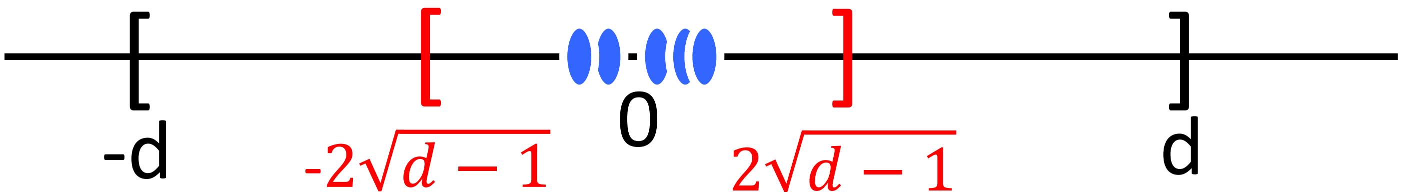
Proof is elementary, doesn't use number theory.

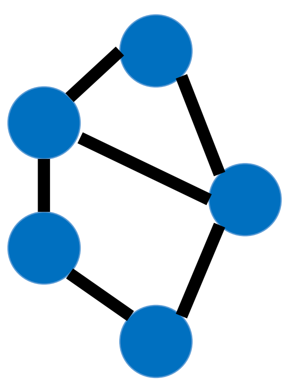
Not explicit.

Based on a new existence argument: method of **interlacing families of polynomials**.

Bilu-Linial'06 Approach

Find an operation which doubles the size of a graph without blowing up its eigenvalues.





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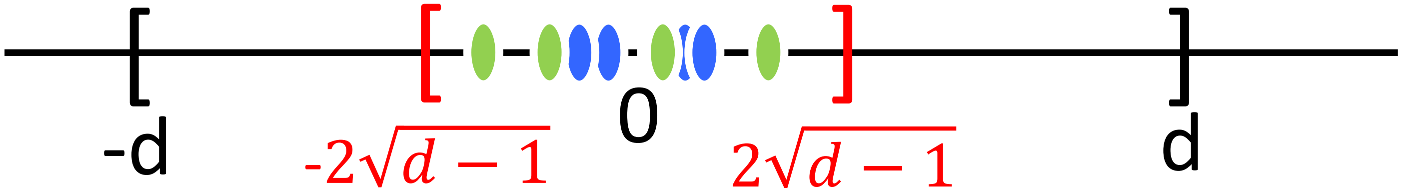

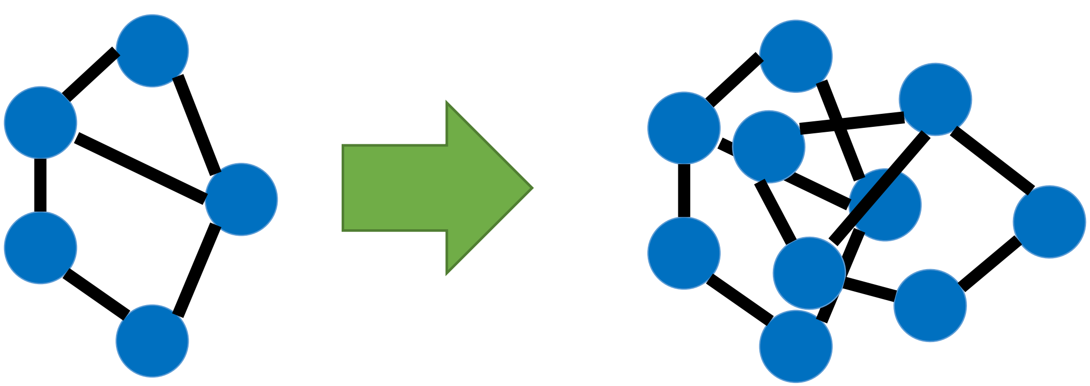

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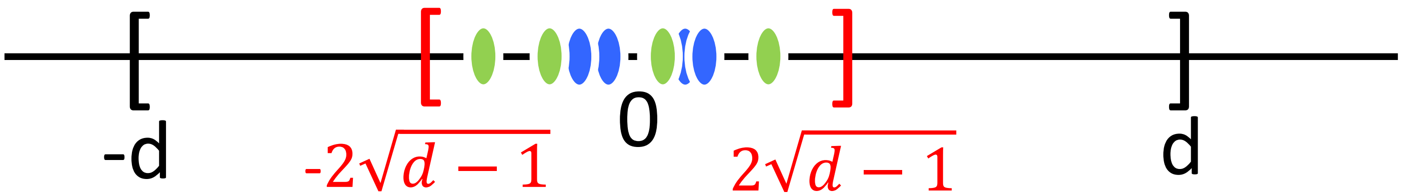
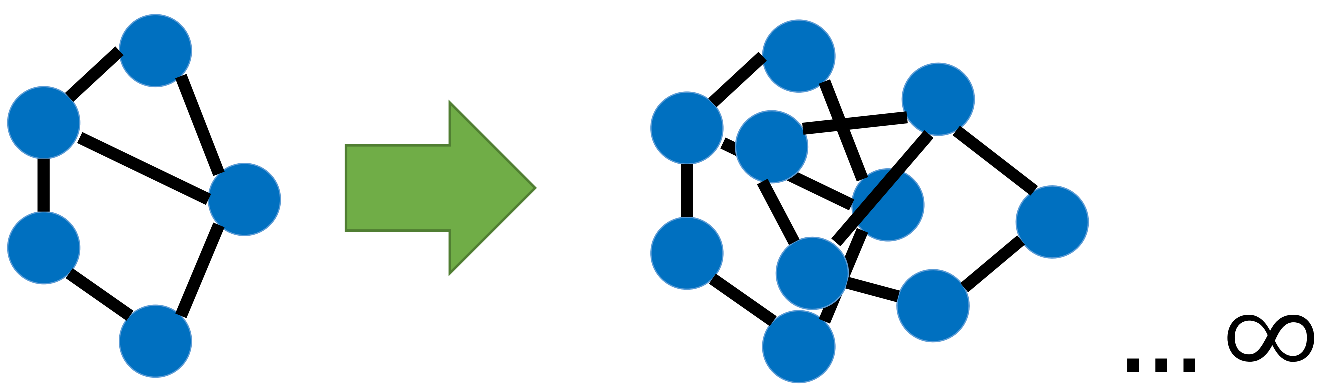
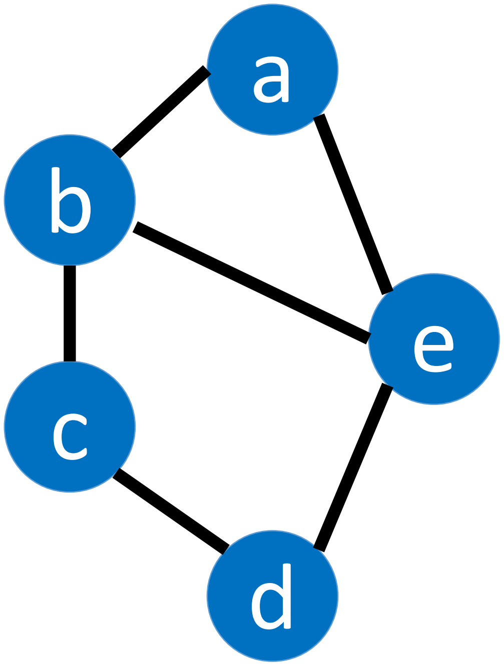




2-lifts of graphs



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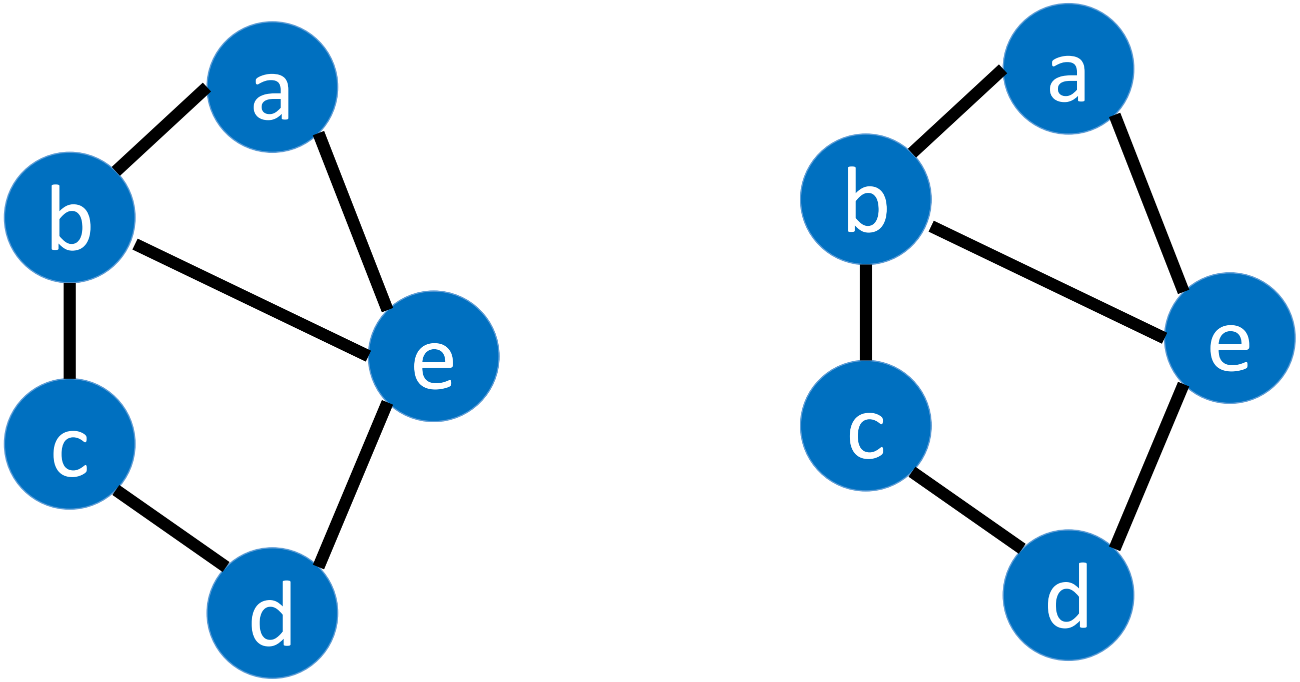

duplicate every vertex

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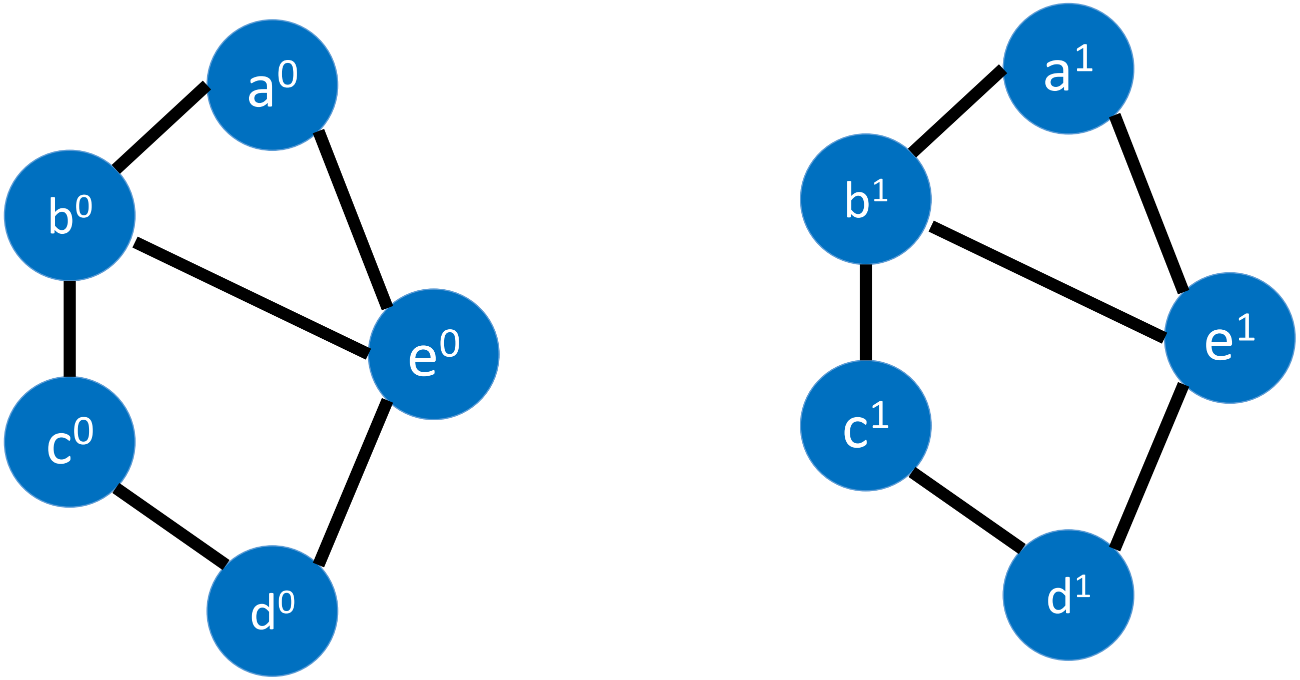

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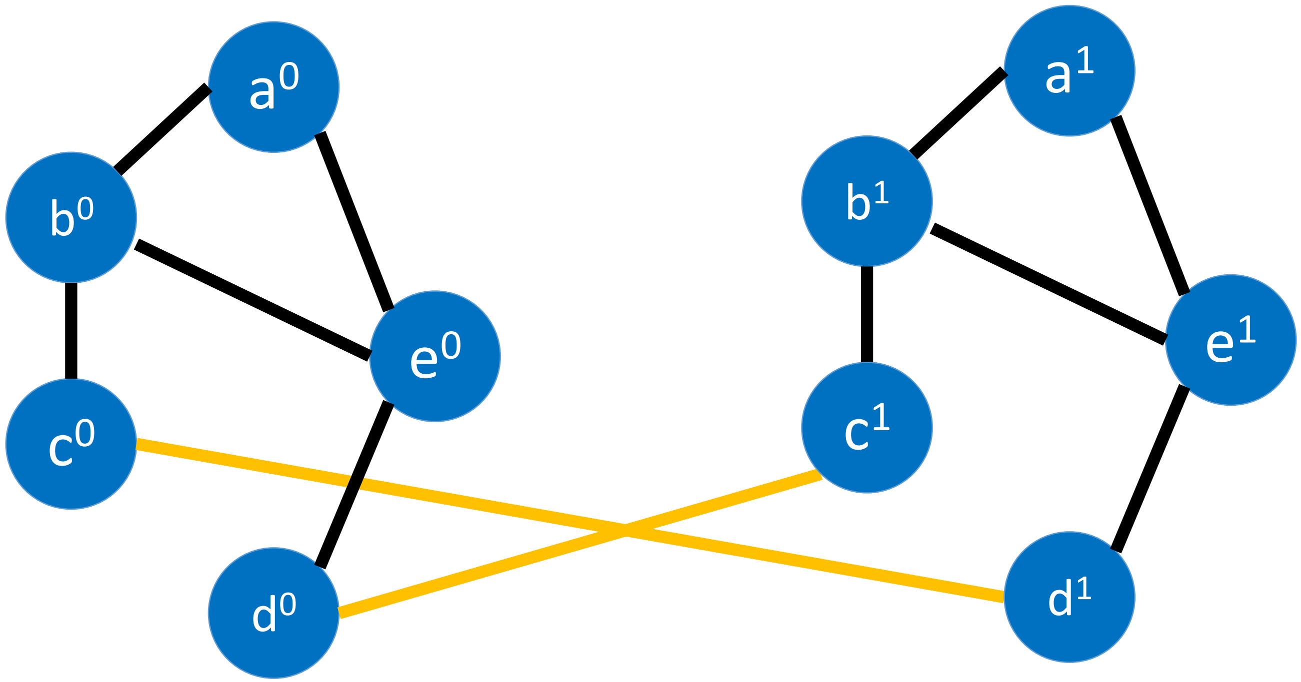

for every pair of edges:
leave on either side (parallel),
or make both cross

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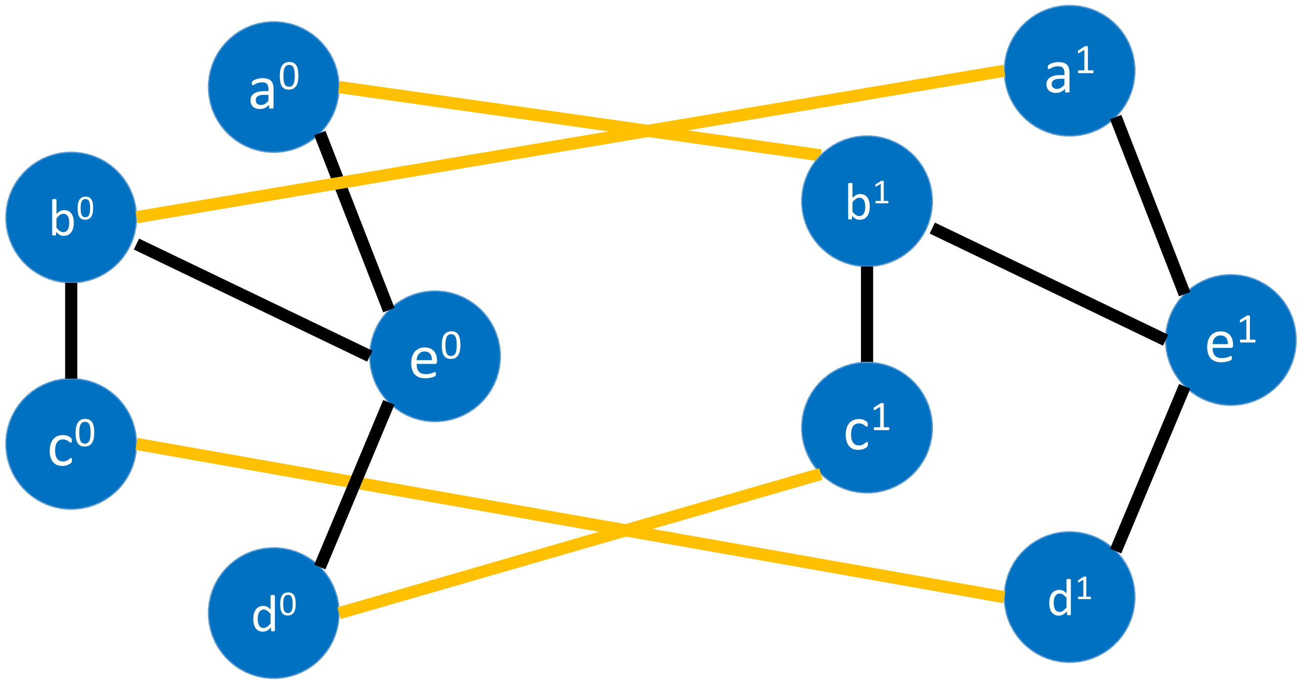

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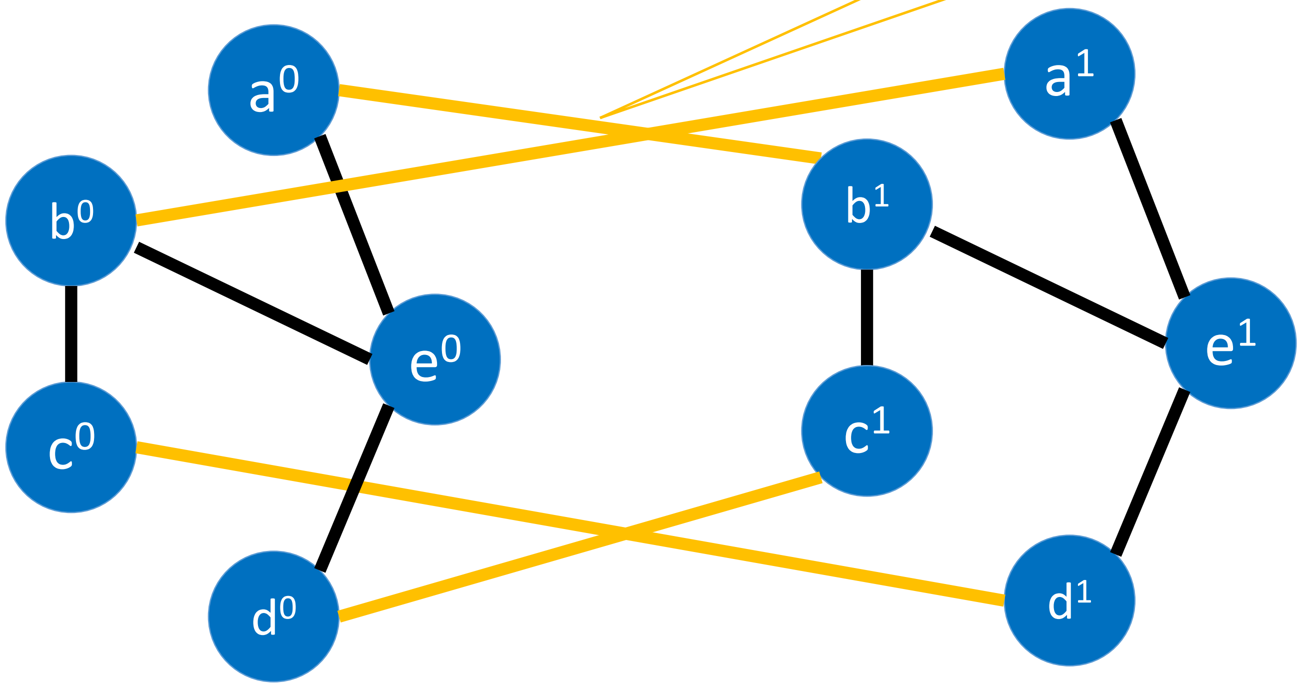
2-lifts of graphs

2^m possibilities



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2-lifts of graphs

0	1	0	0	1
1	0	1	0	1
0	1	0	1	0
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n eigenvalues $\{\lambda_1 \ldots \lambda_n\}$

2-lifts of graphs

$$
\begin{matrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
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0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{matrix}
$$

2-lifts of graphs

0	0	0	0	1	0	1	0	0	0
0	0	1	0	1	1	0	0	0	0
0	1	0	0	0	0	0	0	1	0
0	0	0	0	1	0	0	1	0	0
1	1	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1
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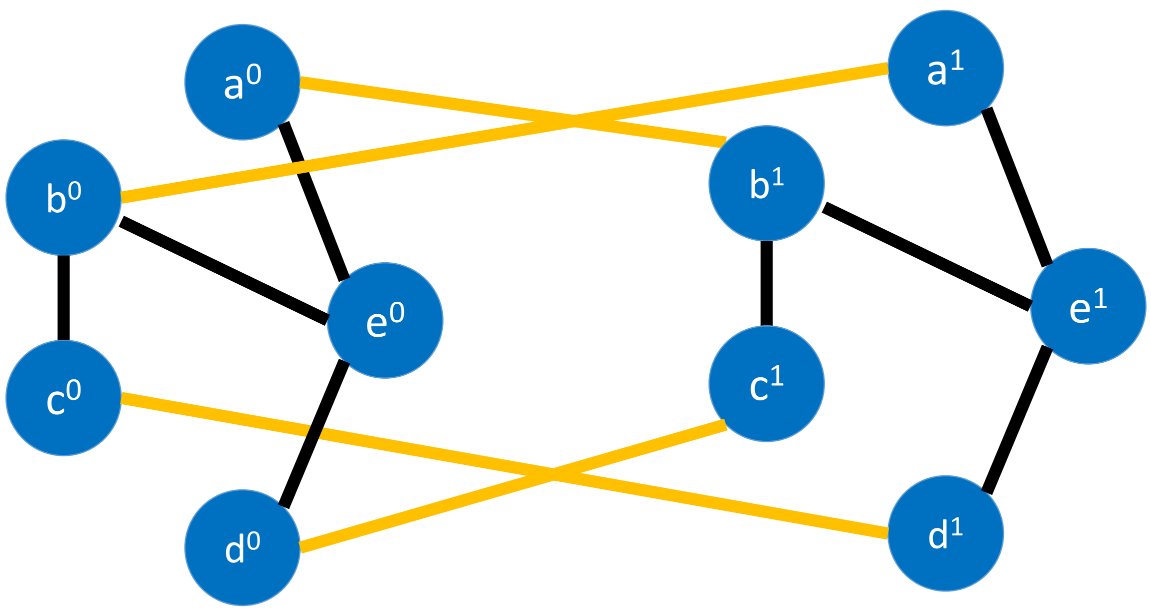
\end{matrix}$$

2n eigenvalues $\{\lambda_1 \ldots \lambda_n\} \cup \{\lambda'_1 \ldots \lambda'_n\}$

Eigenvalues of 2-lifts (Bilu-Linial)

Given a 2-lift of G,
create a signed adjacency matrix A_s
with a -1 for crossing edges
and a 1 for parallel edges



0	-1	0	0	1
-1	0	1	0	1
0	1	0	-1	0
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Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:

The eigenvalues of the 2-lift are the union of the eigenvalues of A (old) and the eigenvalues of A_s (new)

$$\{\lambda_1' \ldots \lambda_n'\} = eigs(A_s)$$

$$A_s = \begin{matrix} 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{matrix}$$

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Conjecture:

Every d-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$

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We prove this in the bipartite case.

Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
Every d-regular adjacency matrix A has a signing A_s with $\lambda_1(A_S) \leq 2\sqrt{d-1}$

Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
Every d-regular **bipartite** adjacency matrix A has a signing A_s with $||A_S|| \leq 2\sqrt{d-1}$

Trick: eigenvalues of bipartite graphs are symmetric about 0, so only need to bound largest

Random Signings

Idea 1: Choose $s \in \{-1,1\}^m$ randomly.

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Unfortunately,

$$\mathbb{E}\|A_s\| \gg 2\sqrt{d-1}$$

(Bilu-Linial showed $O(\sqrt{d \log^3 d})$ when *A* is nearly Ramanujan)

Random Signings

Idea 2: Observe that $\lambda_1(A_s) = \lambda_{max}(\chi_{A_s})$
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Usually useless, but **not here**!

$\{\chi_{A_s}\}_{s\in\{\pm1\}^m}$ is an *interlacing family.*

$\exists s$ *such that* $\lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})$

3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$.

$$\boxed{\exists s \textit{ such that } \lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})}$$

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$\square$

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Step 2: The expected polynomial

Theorem [Godsil-Gutman'81]

For any graph *G*,

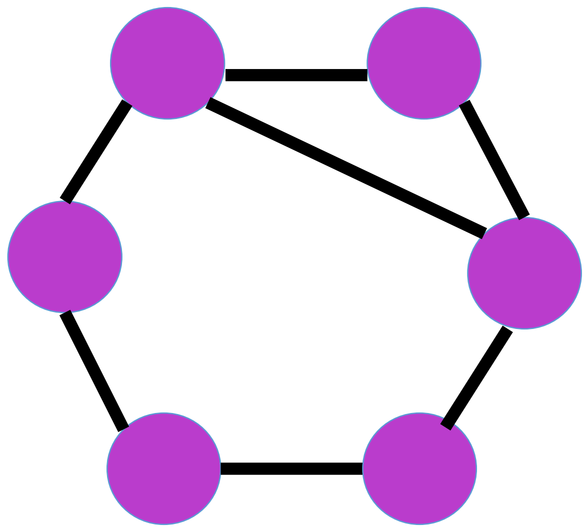
$$\mathop{\mathbb{E}}_{s}\left[\ \chi_{A_s}(x)\ \right] = \mu_G(x)$$

the matching polynomial of G

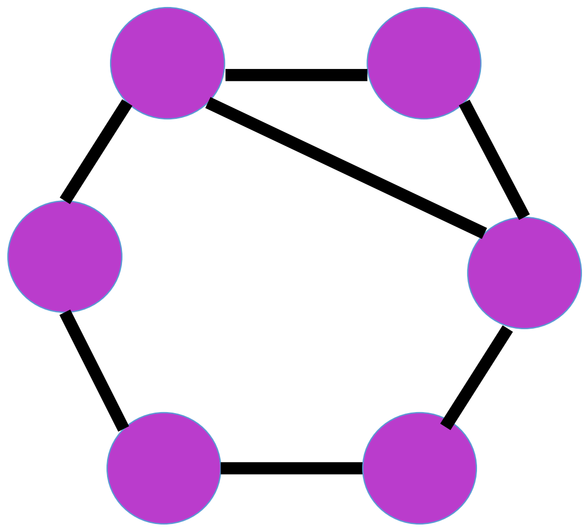
The matching polynomial (Heilmann-Lieb '72)

$$\mu_G(x) = \sum_{i \geq 0} x^{n-2i}(-1)^i m_i$$

m_i = the number of matchings with i edges



$$\mu_G(x) = x^6 - 7x^4 + 11x^2 - 2$$



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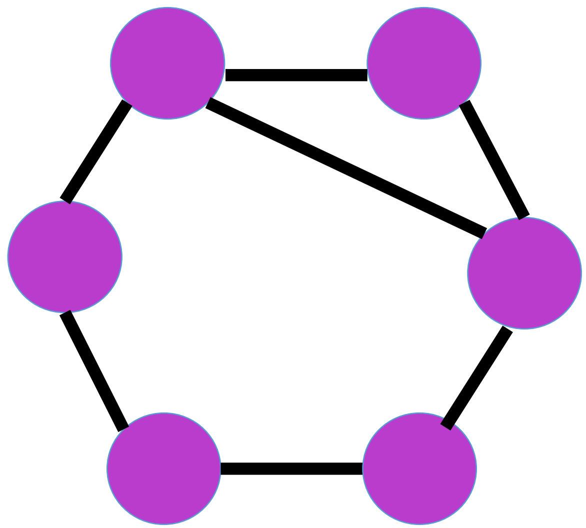
one matching with 0 edges



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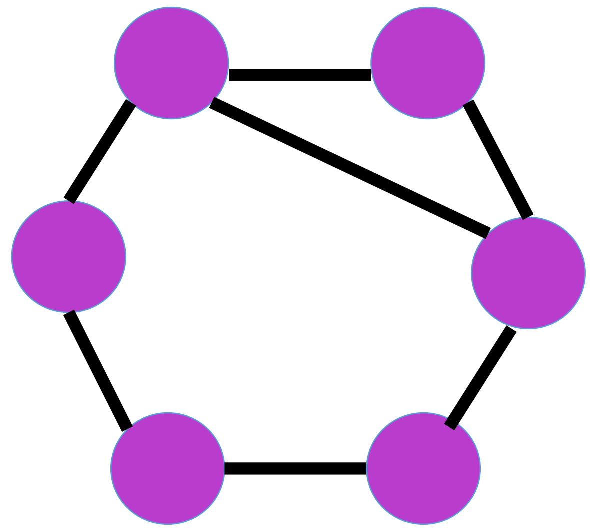
7 matchings with 1 edge



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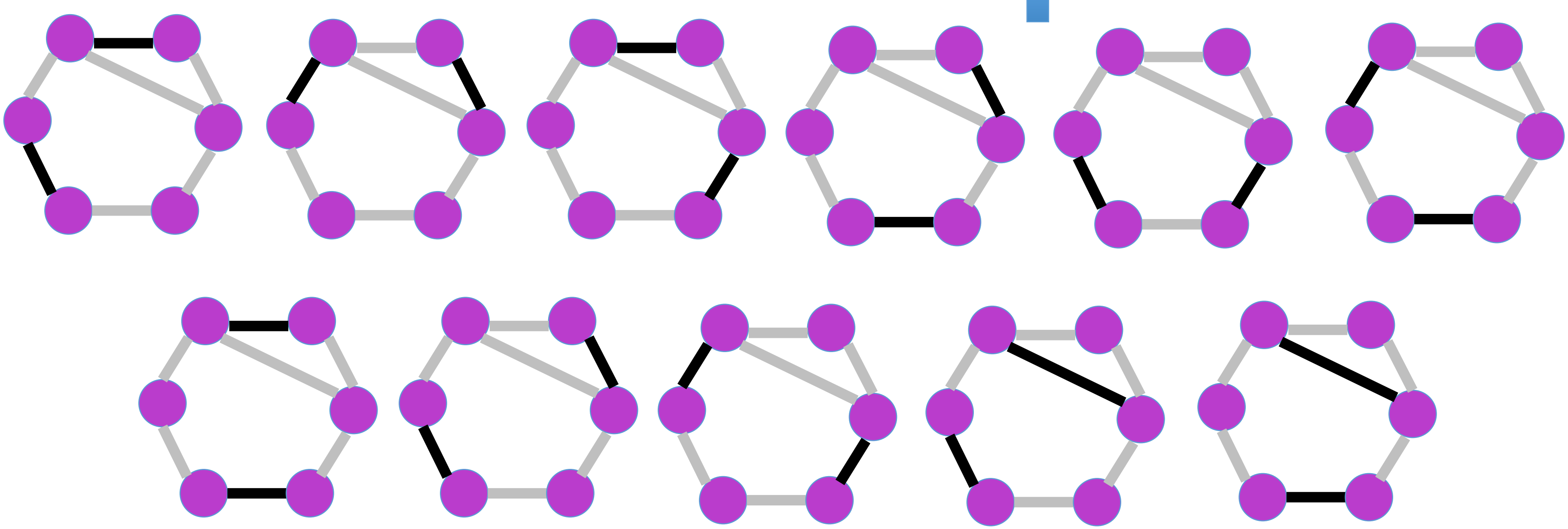

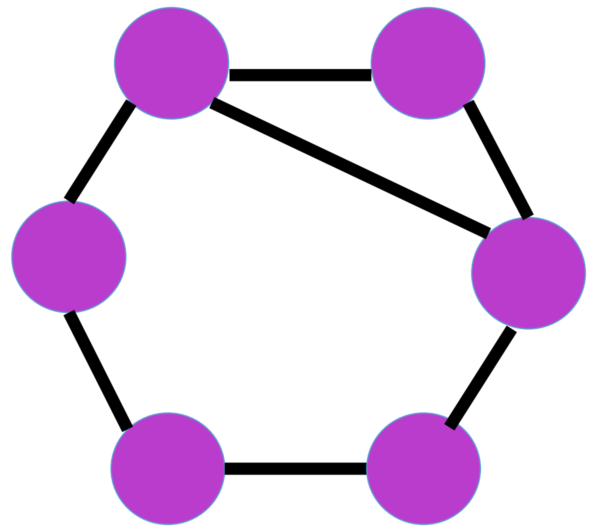

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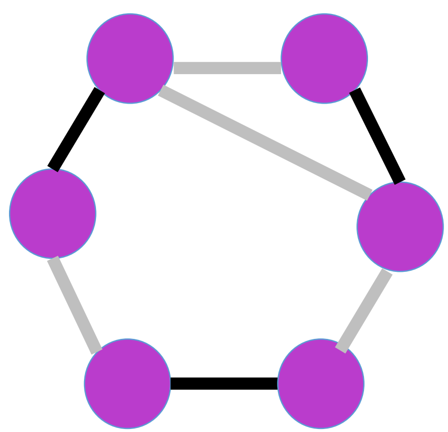

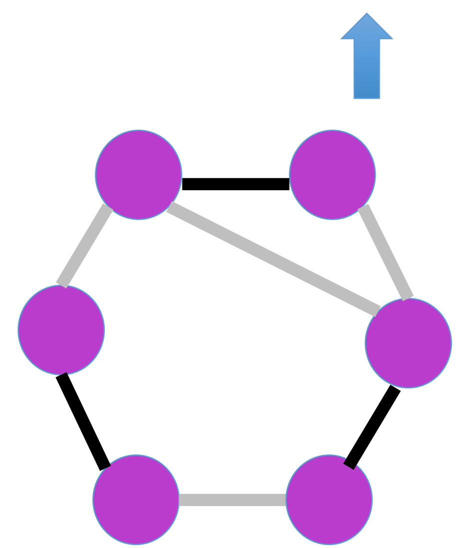

Proof that $\mathbb{E}_{s}[\ \chi_{A_s}(x)\ ] = \mu_G(x)$

Expand $\mathbb{E}_{s}[\ \det(xI - A_s)\ ]$ using permutations

$$\begin{matrix} x & \pm 1 & 0 & 0 & \pm 1 & \pm 1 \\ \pm 1 & x & \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & x & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & x & \pm 1 & 0 \\ \pm 1 & 0 & 0 & \pm 1 & x & \pm 1 \\ \pm 1 & 0 & 0 & 0 & \pm 1 & x \end{matrix}$$

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same value

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Get 0 if hit any 0s

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Get 0 if take just one entry for any edge

Proof that $\mathop{\mathbb{E}}_{s} [\ \chi_{A_s}(x)\ ] = \mu_G(x)$

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$$

Only permutations that count are involutions

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Correspond to matchings

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$$
\begin{matrix}
x & \pm 1 & 0 & 0 & \pm 1 & \pm 1 \\
\pm 1 & x & \pm 1 & 0 & 0 & 0 \\
0 & \pm 1 & x & \pm 1 & 0 & 0 \\
0 & 0 & \pm 1 & x & \pm 1 & 0 \\
\pm 1 & 0 & 0 & \pm 1 & x & \pm 1 \\
\pm 1 & 0 & 0 & 0 & \pm 1 & x
\end{matrix}
$$

Only permutations that count are involutions

Correspond to matchings

$\square$

3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$.

$\exists s$ *such that* $\lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})$

$\boxed{\begin{array}{l}\textbf{2. } \text{Calculate the expected polynomial.} \\ \mathbb{E}\chi_{A_s}(x) = \mu_G(x) \qquad \textbf{[Godsil-Gutman'81]}\end{array}}$ $\checkmark$

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$\lambda_{max}(\mu_G(x)) \leq 2\sqrt{d-1}$

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$$\mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i$$

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Theorem (Heilmann-Lieb)
all the roots are real
and have absolute value at most $2\sqrt{d-1}$

Proof: simple, based on recurrences.

$\square$

3-Step Proof Strategy

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Implied by:

“ $\{\chi_{A_s}\}_{s\in\{\pm 1\}^m}$ is an interlacing family.”

Averaging Polynomials

Basic Question: Given p_0, p_1 when are the roots of the $p_i(x)$ related to roots of $\mathbb{E}_i p_i(x)$?

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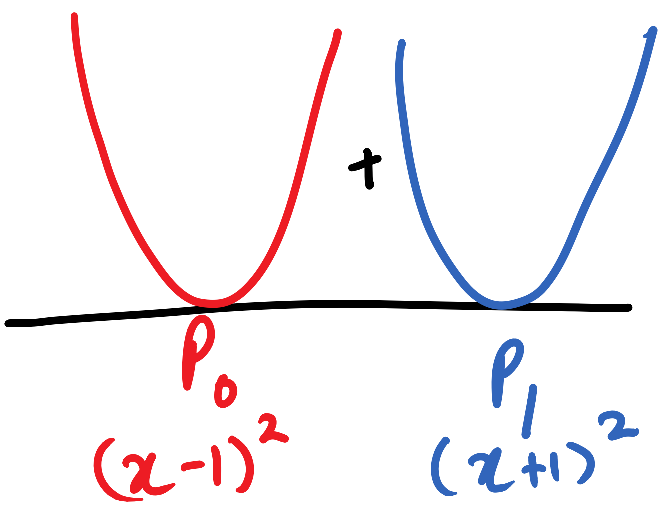
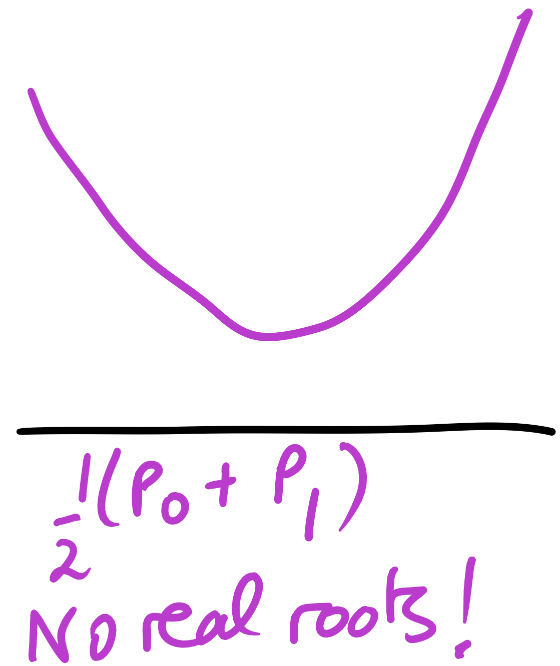
Answer: Certainly not always





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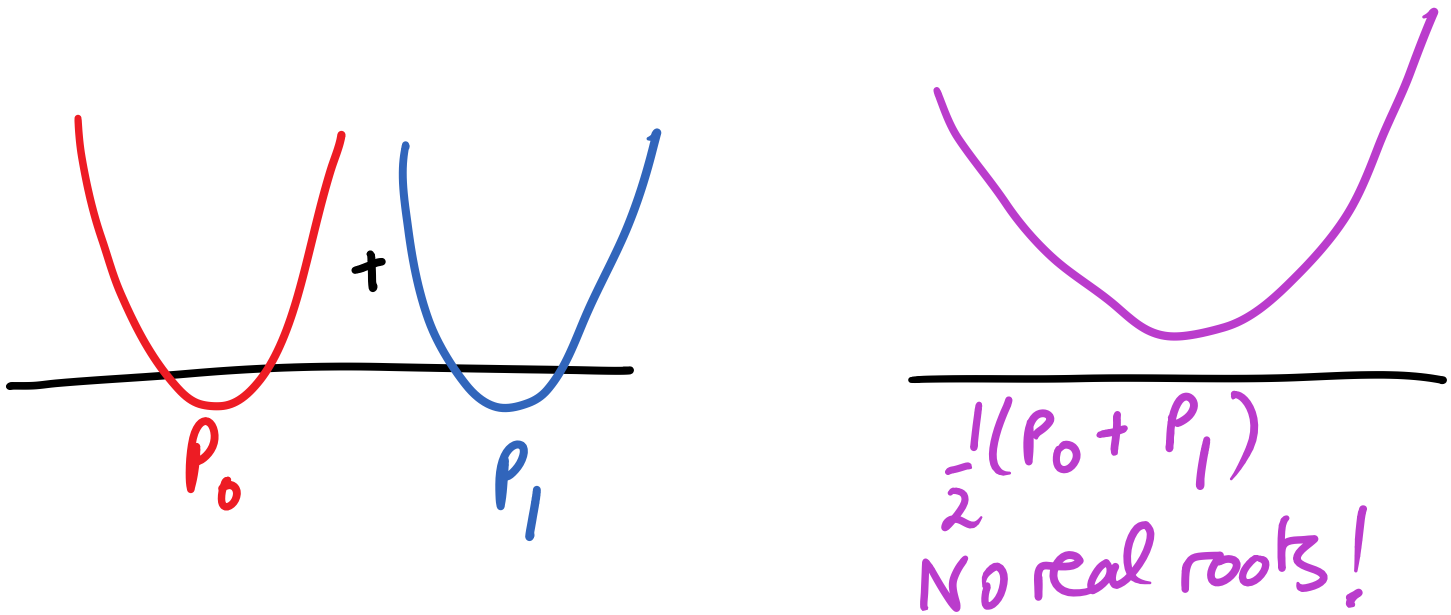

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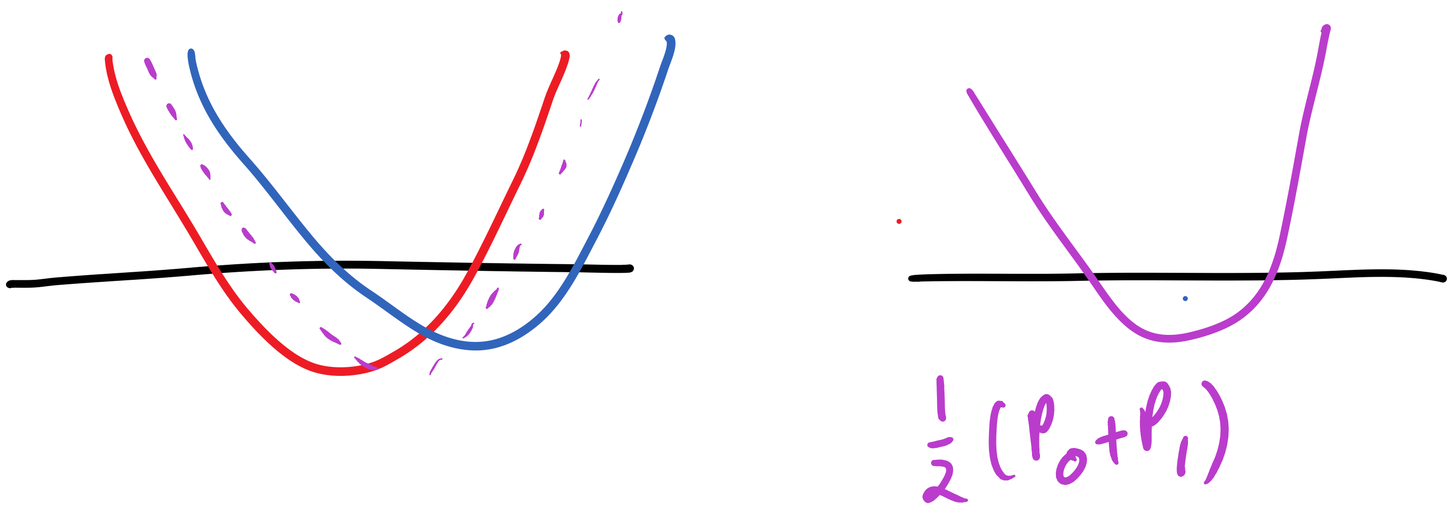


A Sufficient Condition

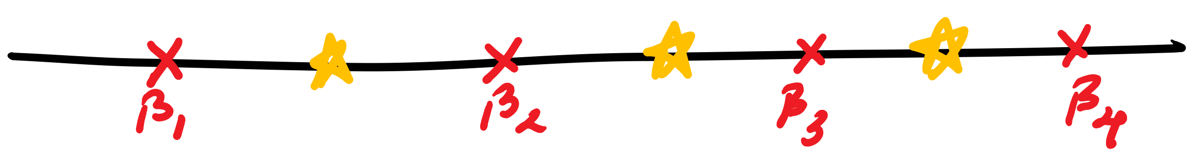
Basic Question: Given p_0, p_1 when are the roots of the $p_i(x)$ related to roots of $\mathbb{E}_i p_i(x)$?

Answer: When they have a *common interlacing*.

Definition. $q = \prod_{i=1}^{n-1}(x - \alpha_i)$ *interlaces*

$p = \prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_n \leq \alpha_{n-1} \leq \beta_{n-1} \ldots \leq \alpha_1 \leq \beta_1.$$



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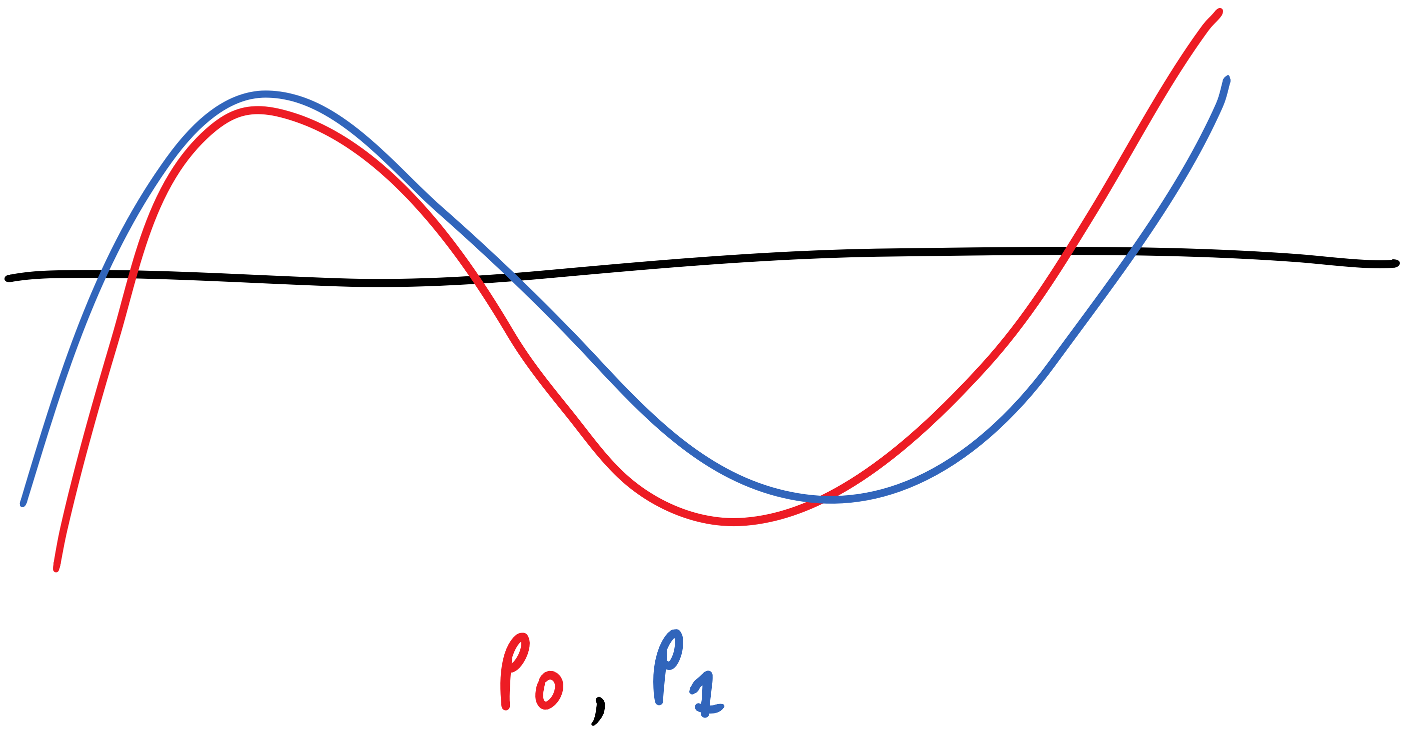

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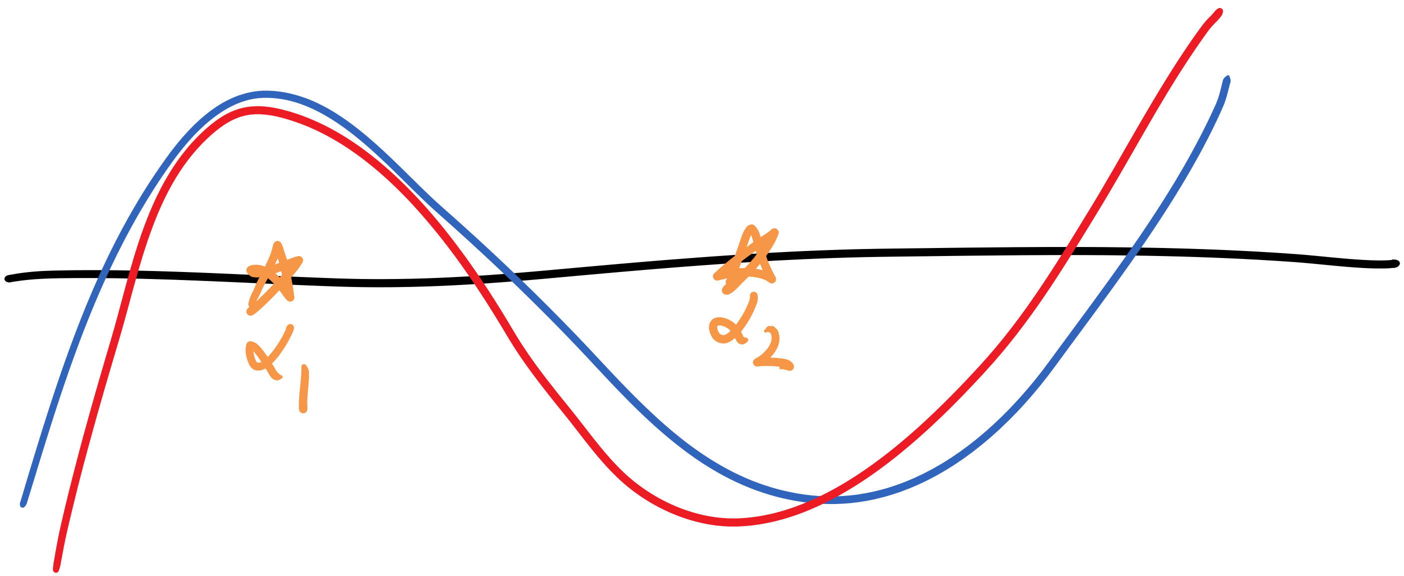

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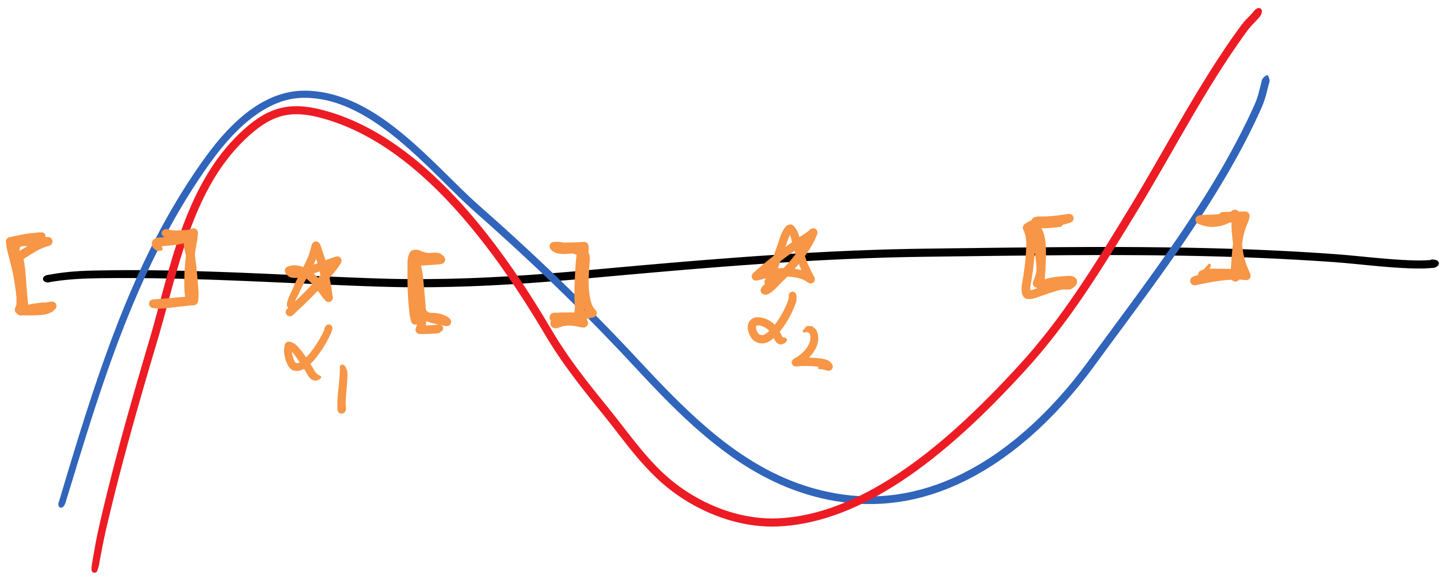

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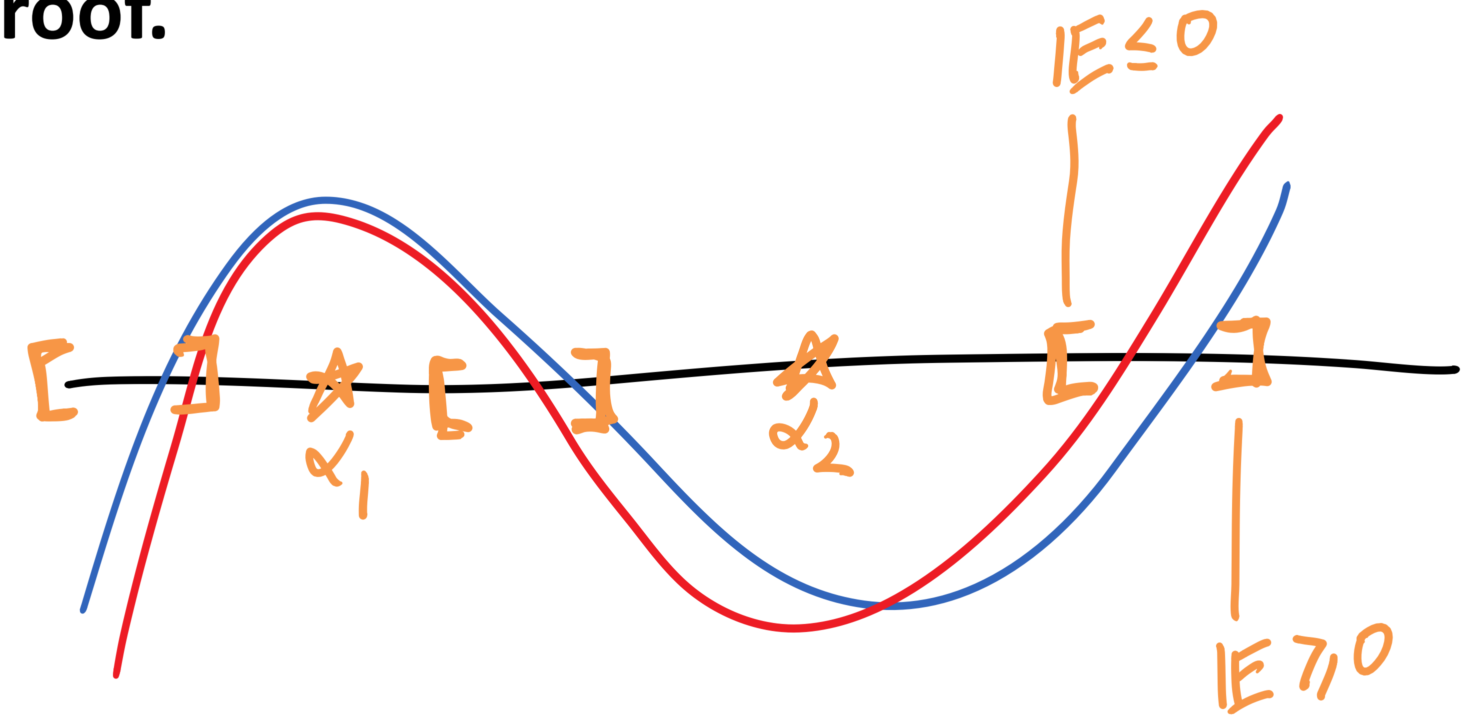

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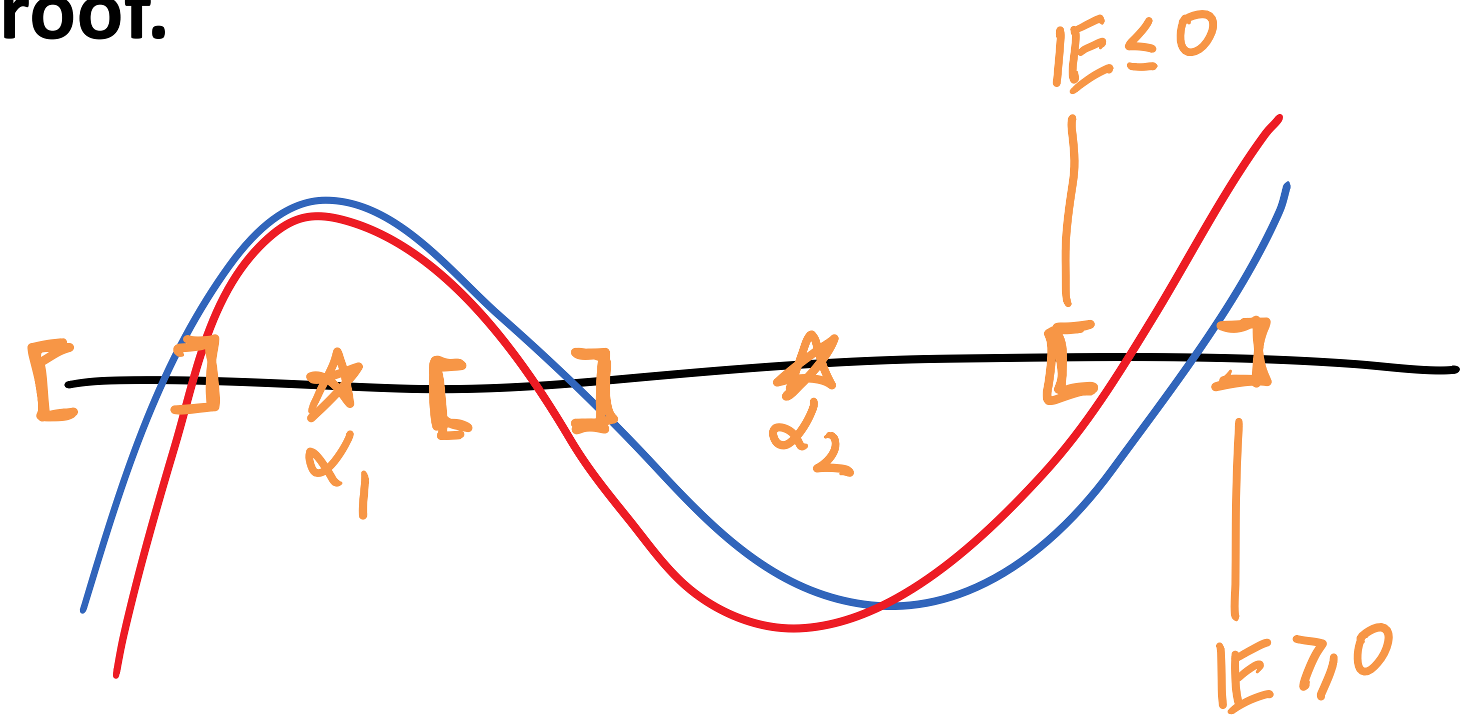

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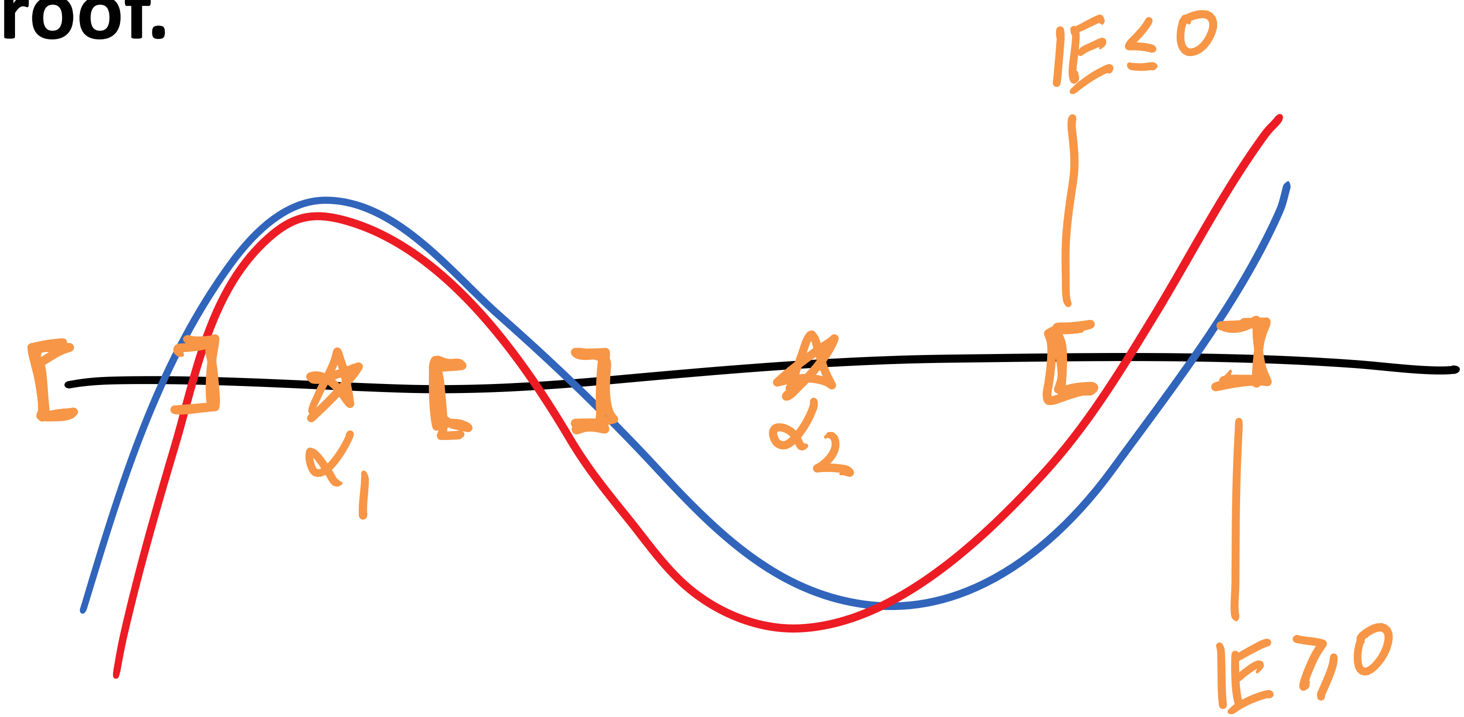
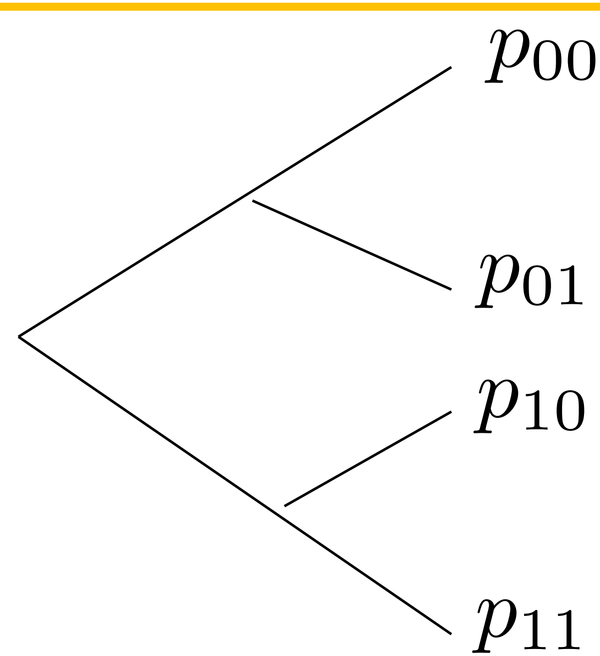


$$\text{So } \lambda_{max}(p_0) \leq \lambda_{max}(\mathbb{E}p_i)$$

$\square$

Interlacing Family of Polynomials

Definition: $\{p_s\}_{s \in \{0,1\}^m}$ is an interlacing family if can be placed on the leaves of a tree so that when every node is the sum of leaves below, sets of siblings have common interlacings



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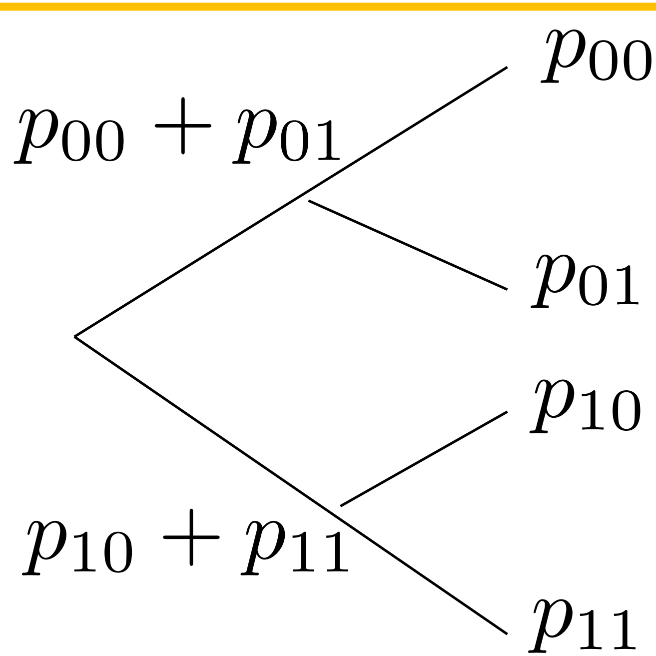

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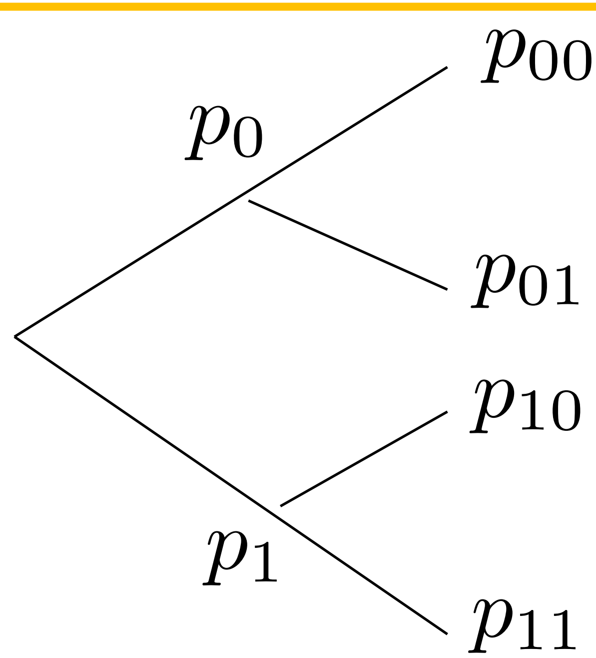

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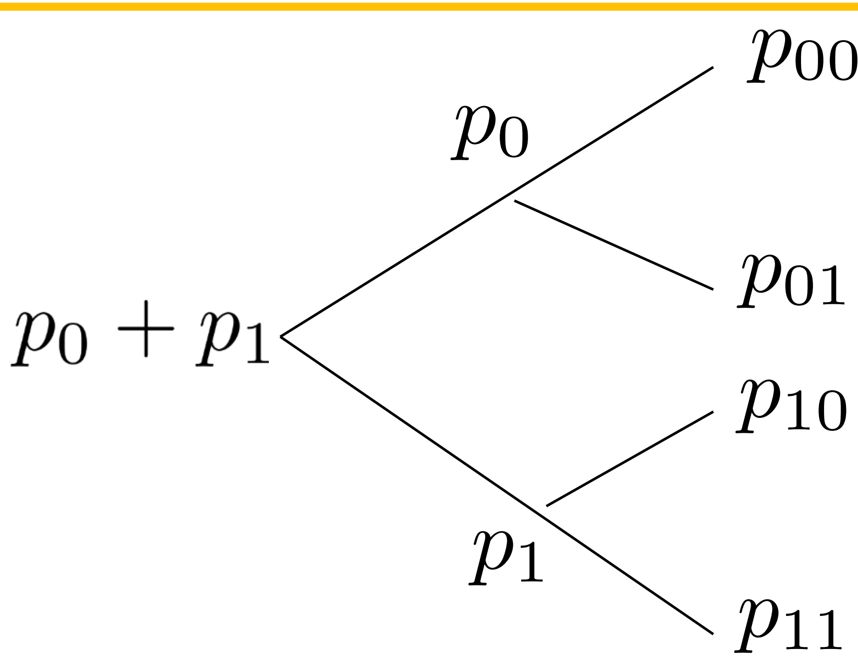

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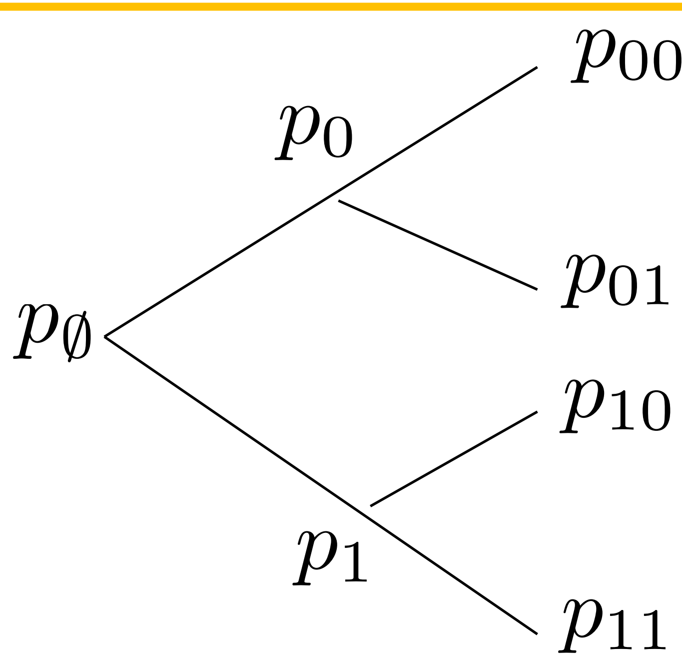

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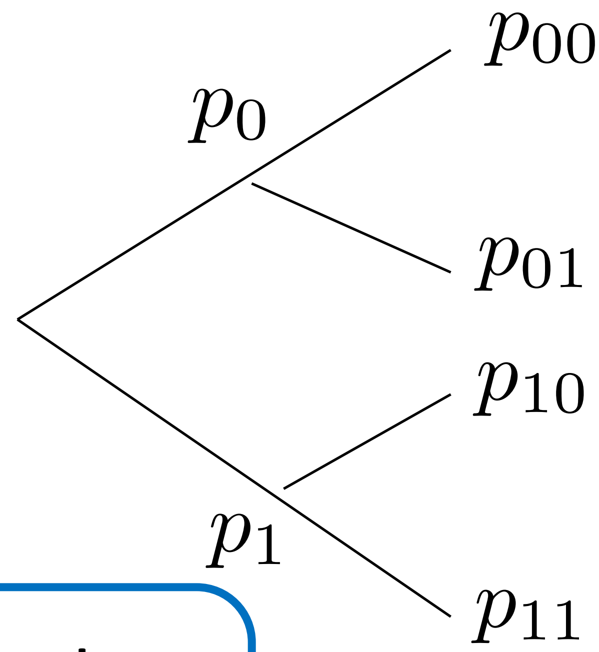

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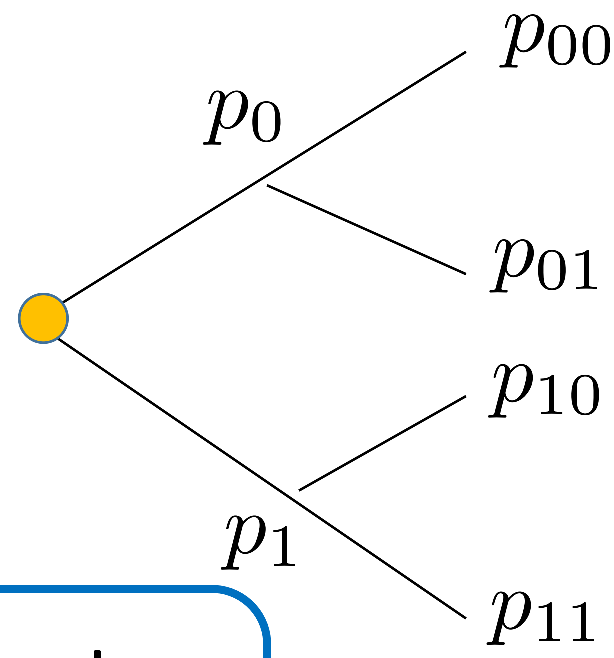

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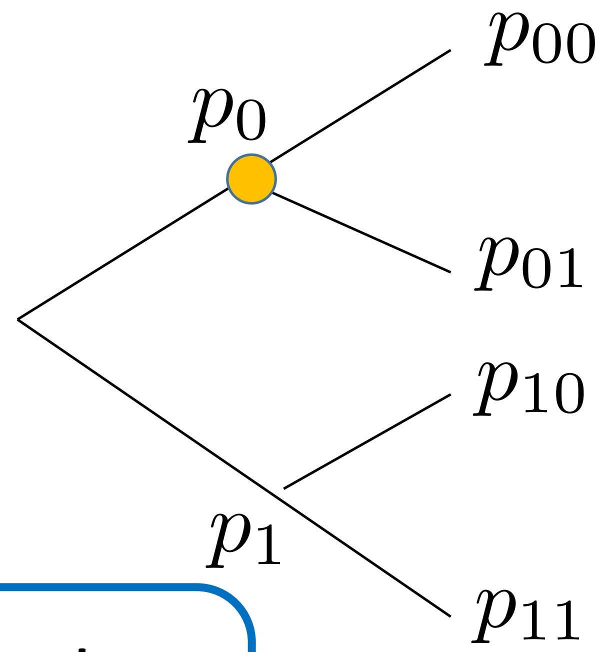

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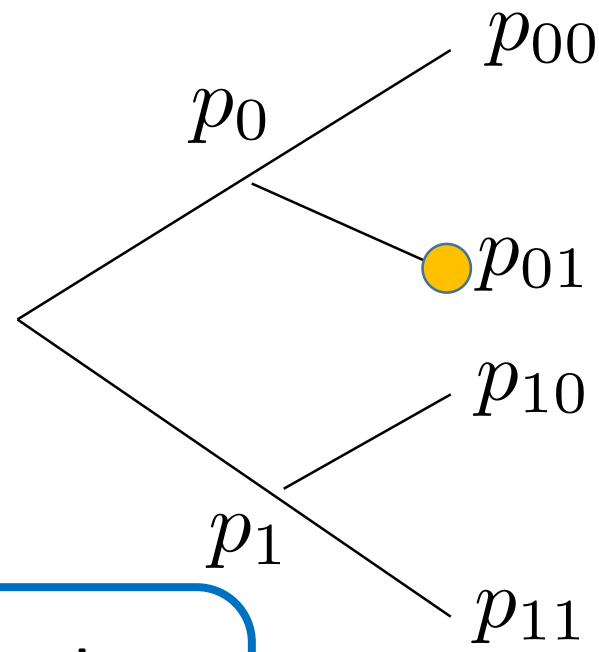

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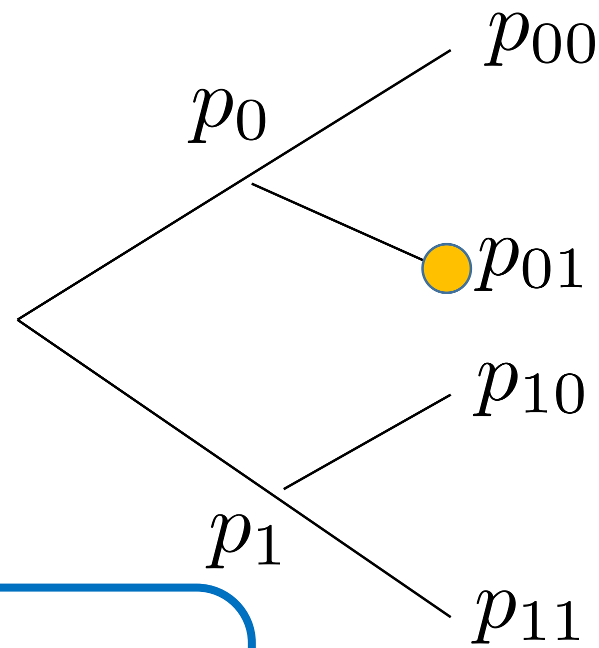

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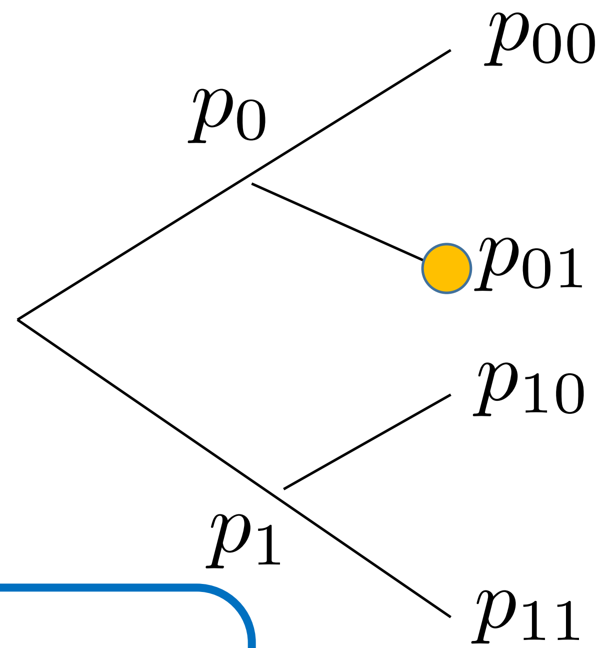

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Lemma (easy):

$p_0(x)$ and $p_1(x)$ have a common interlacing

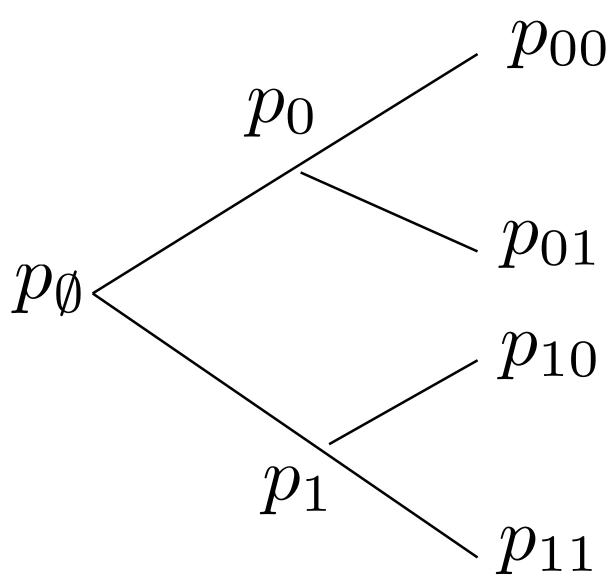
if and only if $\lambda p_0(x) + (1-\lambda)p_1(x)$

is real rooted for all $0 \leq \lambda \leq 1$

To prove interlacing family

Let $p_{s_1,...,s_k}(x) = \mathbb{E}_{s_{k+1},...,s_m} [\, p_{s_1,...,s_m}(x) \,]$

Leaves of tree = signings $s_1, \ldots, s_m$
Internal nodes = partial signings $s_1, \ldots, s_k$



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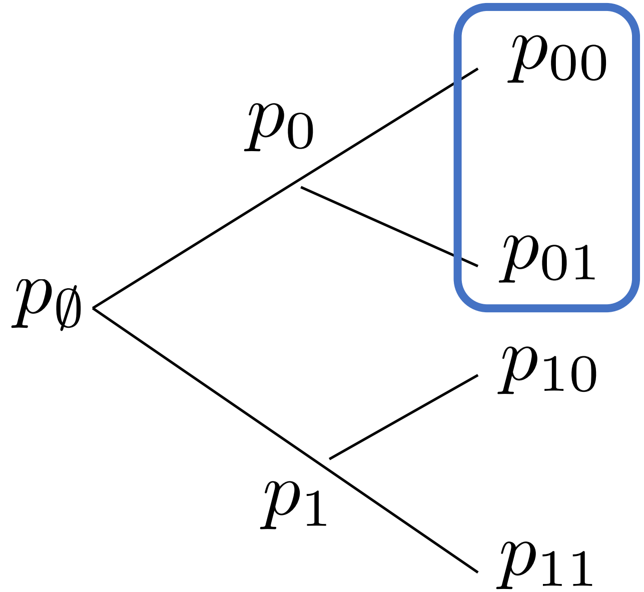

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is real rooted

$s_1, \ldots, s_k$ *are fixed*

s_{k+1} *is 1 with probability* λ, *-1 with* $1-\lambda$

$s_{k+2}, \ldots, s_m$ *are uniformly* ± 1

Generalization of Heilmann-Lieb

Suffices to prove that

$$\mathbb{E}_{s \in \{\pm 1\}^m} \left[\ p_s(x)\ \right] \quad \text{is real rooted}$$

for **every** independent distribution on the entries of *s*

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$$\sum_{s \in \{\pm 1\}^m} p_s(x) \prod_{i : s_i = 1} \lambda_i \prod_{i : s_i = -1} (1 - \lambda_i)$$

$$\lambda_1, \ldots, \lambda_m \in [0, 1]$$

Transformation to PSD Matrices

Suffices to show real rootedness of

$$\mathbb{E}_{s\in\{\pm 1\}^m} p_s(x-d) = \mathbb{E}_{s\in\{\pm 1\}^m} \det(xI - (dI - A_s))$$

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$$\mathbb{E}_s \det\big(xI - (dI - A_s)\big) = \mathbb{E}\det\left(xI - \sum_{ij \in E} v_{ij} v_{ij}^T\right)$$

where $v_{ij} = \begin{cases} (\delta_i - \delta_j) \text{ with probability } \lambda_{ij} \\ (\delta_i + \delta_j) \text{ with probability } (1 - \lambda_{ij}) \end{cases}$

Master Real-Rootedness Theorem

Given *any* independent random vectors $v_1, \ldots, v_m \in \mathbb{R}^d$, their expected characteristic polymomial

$$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$$

has real roots.

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How to prove this?

The Multivariate Method

A. Sokal, 90's-2005:

> "…it is often useful to consider the multivariate polynomial … even if one is ultimately interested in a particular one-variable specialization"

Borcea-Branden 2007+: prove that univariate polynomials are real-rooted by showing that they are nice transformations of *real-rooted multivariate polynomials*.

Real Stable Polynomials

Definition: $p \in \mathbb{R}[z_1, \ldots, z_n]$

is *real stable* if $\text{imag}(z_i) > 0$ for all i

Implies $p(z_1, \ldots, z_n) \neq 0$.

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no roots in the upper half-plane

univariate real stable = real-rooted

Excellent Closure Properties

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Implies $p(z_1, \ldots, z_n) \neq 0$.

If $p \in \mathbb{R}[z_1, \ldots, z_n]$ is real stable, then so is

1. $p(\alpha, z_2, \ldots, z_n)$ for any $\alpha \in \mathbb{R}$
2. $(1 - \partial_{z_i})p(z_1, \ldots z_n)$

A Useful Real Stable Poly

Borcea-Brändén ‘08:

For PSD matrices $A_1, \ldots, A_k$

$$\det(\textstyle\sum_i z_i A_i)$$

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Plan: apply closure properties to this

to show that $\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$ is real stable.

Central Identity

Suppose $v_1, \dots, v_m$ are **independent** random vectors with $A_i := \mathbb{E} v_i v_i^T$. Then

$$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$$
$$= \prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right)\det\left(xI + \sum_i z_i A_i\right)\Bigg|_{z_1=\cdots=z_m=0}$$

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Proof: easy, tomorrow. $\square$

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Real Stable

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Suppose $v_1, \ldots, v_m$ are **independent** random vectors with $A_i := \mathbb{E} v_i v_i^T$. Then

$$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$$

$$= \prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right)\det\left(xI + \sum_i z_i A_i\right)\Bigg|_{z_1=\cdots=z_m=0}$$

Real Stable: Closure under $(1-\partial)$

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Real stable: closure under $z_i = 0$

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Real Stable ⇒ Real Rooted!

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□

The Whole Proof

$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$ is real-rooted for all indep. v_i.

The Whole Proof

$\mathbb{E}\chi_{A_s}(d - x)$ is real-rooted for all product distributions on signings.

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The Whole Proof

$\boxed{\{\chi_{A_s}(x)\}_{x \in \{\pm 1\}^m} \text{ is an interlacing family}}$

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The Whole Proof

$$\boxed{\exists s \text{ such that } \lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})}$$

$\{\chi_{A_s}(x)\}_{x \in \{\pm 1\}^m}$ is an interlacing family

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3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$. $\surd$

$$\boxed{\exists s \text{ such that } \lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})}$$

2. Calculate the expected polynomial. $\surd$

$$\mathbb{E}\chi_{A_s}(x) = \mu_G(x)$$

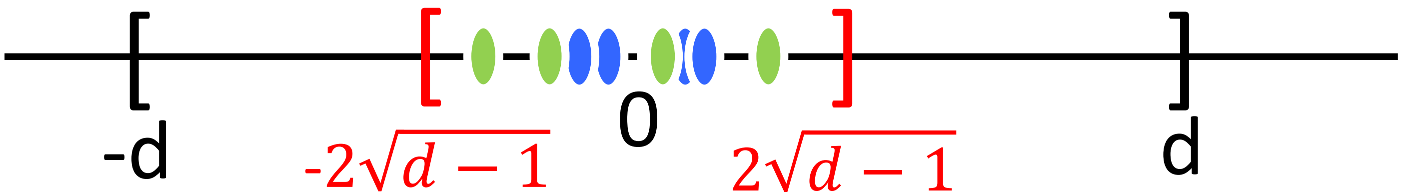
3. Bound the largest root of the expected poly. $\surd$

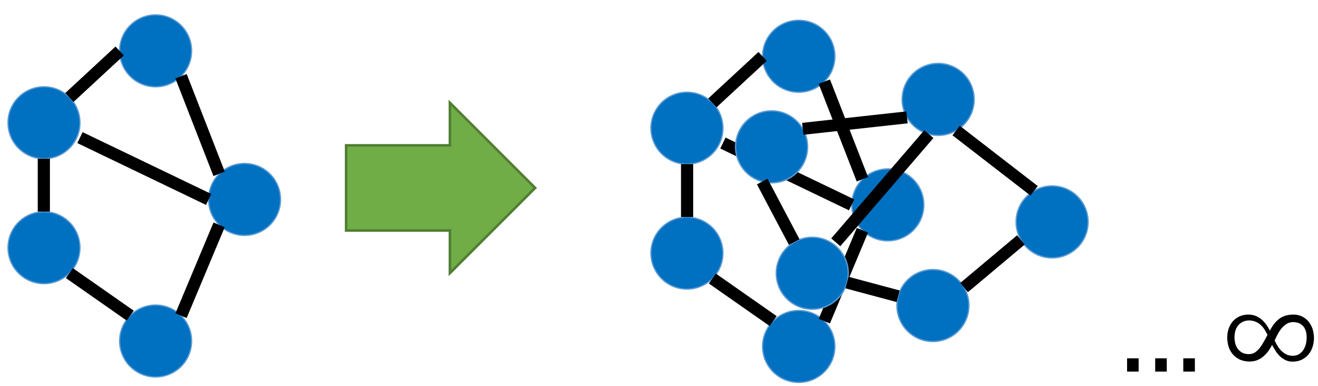
$$\lambda_{max}(\mu_G(x)) \leq 2\sqrt{d-1}$$

$\square$

Infinite Sequences of Bipartite Ramanujan Graphs

Find an operation which doubles the size of a graph without blowing up its eigenvalues.





Main Theme

Reduced the existence of a good matrix to:

1. Proving real-rootedness of an expected polynomial.

2. Bounding roots of the expected polynomial.

Main Theme

Reduced the existence of a good matrix to:

1. Proving real-rootedness of an expected polynomial.
   (general, using real stability)
2. Bounding roots of the expected polynomial.
   (specific, using combinatorics)

Tomorrow

Reduced the existence of a good matrix to:

1. Proving real-rootedness of an expected polynomial.
   (general, using real stability)
2. Bounding roots of the expected polynomial.
   (**general, using new method**)

Tom

major implications in combinatorics, linear algebra + 1959 Kadison-Singer Conjecture.

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