# Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem

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# Discrepancy Theory

"How well can you approximate a discrete object by a continuous one."

e.g. Given a set family $S_1, \ldots, S_m \subset [n]$, find a red-blue coloring $[n] = R \cup B$ such that every set is half red and half blue.

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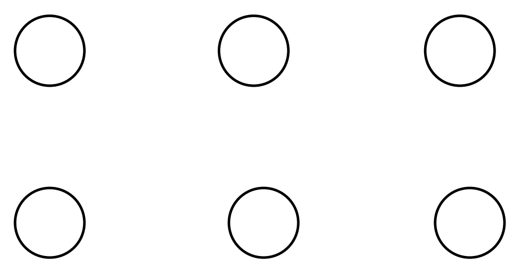

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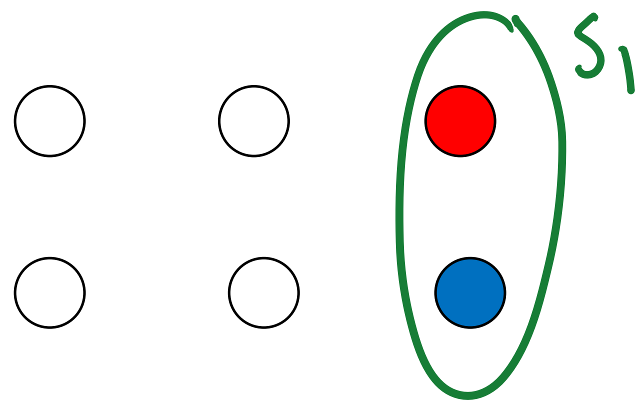

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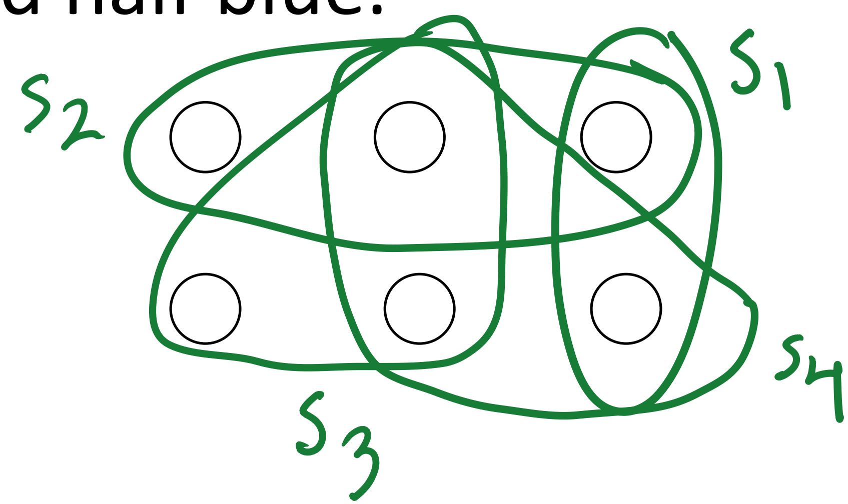

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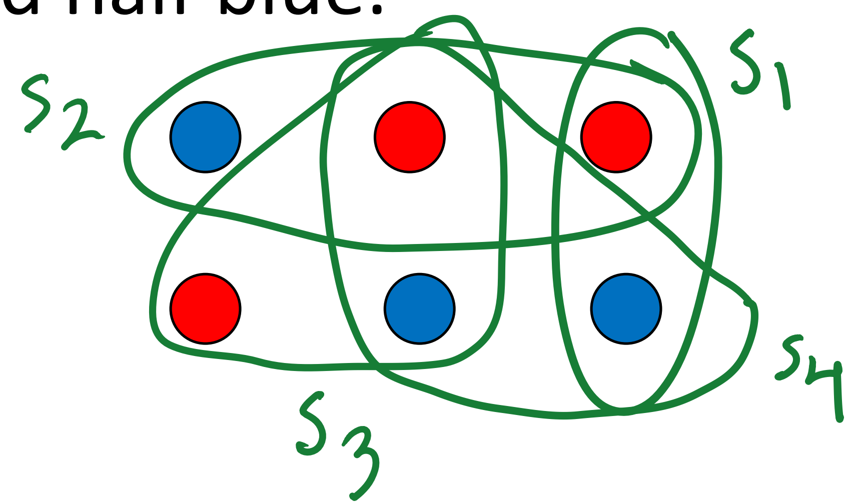

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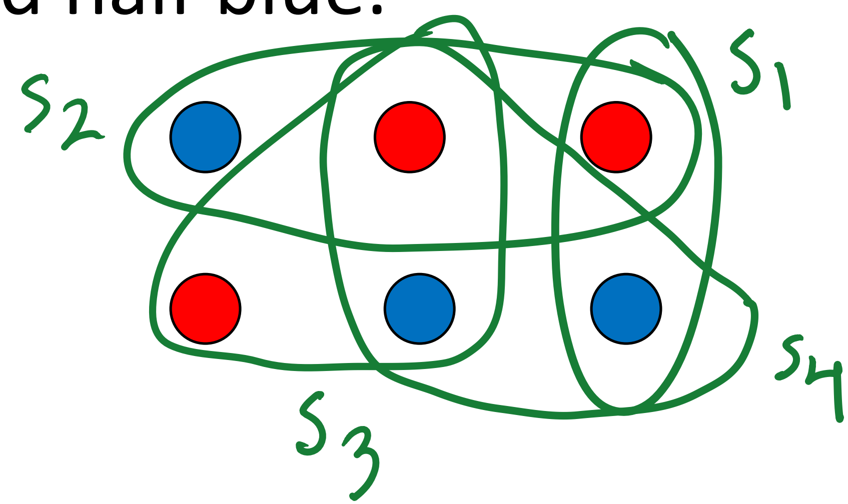
$$disc = 1$$



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How well can you do in general?

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In general, a random coloring gives

$$disc := \max_i \left| |S_i \cap R| - \frac{|S_i|}{2} \right| \leq O\left(\sqrt{n \log m}\right).$$

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**Spencer**: There exists a coloring with

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# A Spectral Discrepancy Theorem

# Quadratic Forms and Energy

Given vectors $v_1, \ldots, v_m \in \boldsymbol{R}^n$, their **energy** in a test direction $u \in \boldsymbol{R}^n$, $||u|| = 1$, is the quadratic form

$$Q(u) \coloneqq \sum_i \langle u, v_i \rangle^2$$

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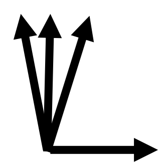

$$v_1, v_2, v_3, v_4 \in \boldsymbol{R}^2$$

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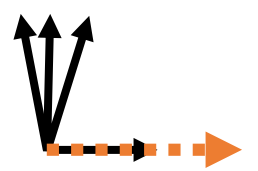

$u = (1,0)$

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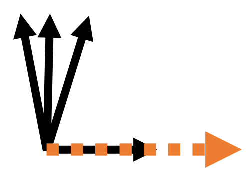

$u = (1,0)$

$\langle u, v_1 \rangle^2 + \langle u, v_2 \rangle^2$
$+\langle u, v_3 \rangle^2 + \langle u, v_4 \rangle^2$
=1 + 0 + ¼ + ¼
=1.5

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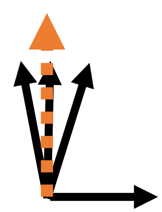

$u = (0,1)$

$\langle u, v_1 \rangle^2 + \langle u, v_2 \rangle^2$
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=0 + 1 + ¾ + ¾
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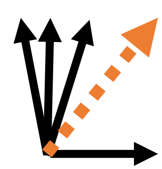

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=2

$$u = (1/\sqrt{2}, 1/\sqrt{2})$$

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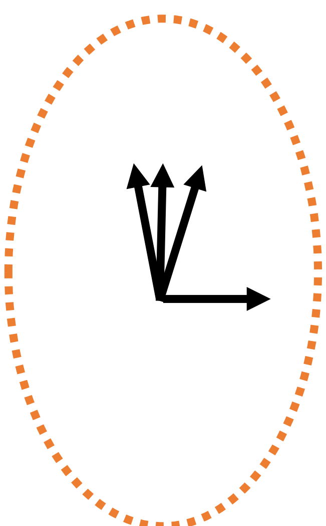

# Splitting A Quadratic Form in Half

**Main Theorem**. Suppose $v_1, \ldots, v_m \in \boldsymbol{R}^n$ are vectors $||v_i|| \le \epsilon$ and energy one in each direction:

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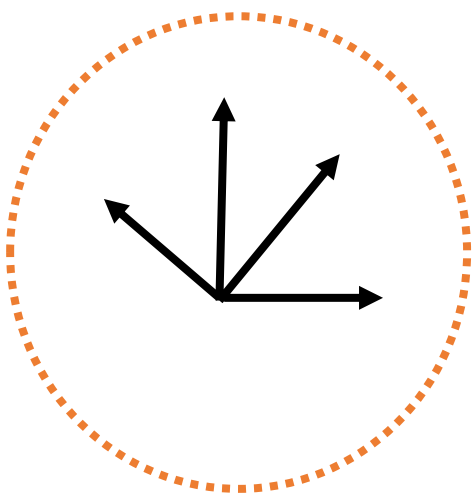

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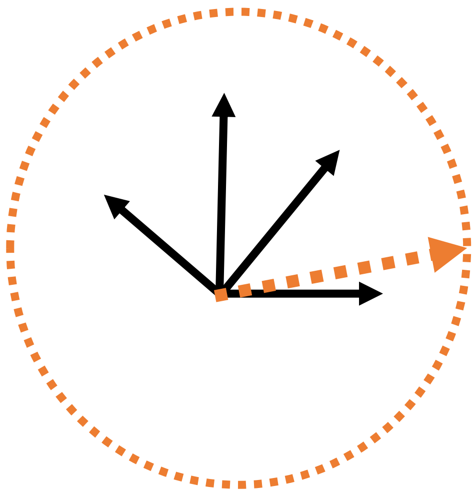

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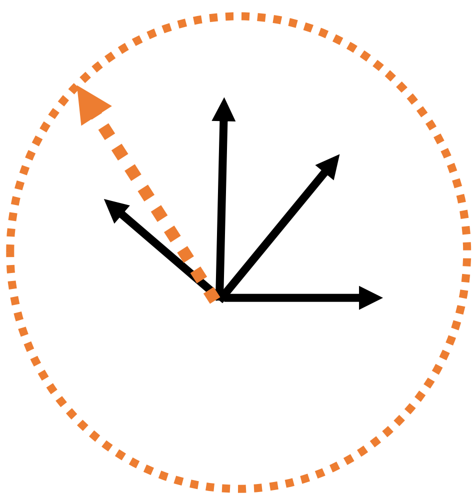

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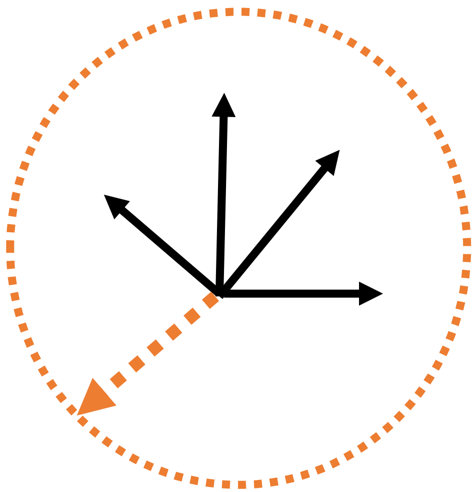

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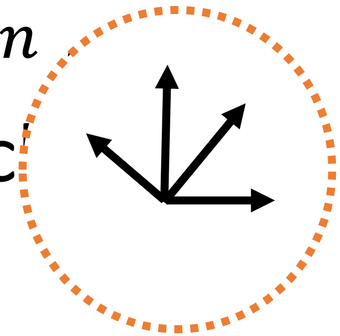

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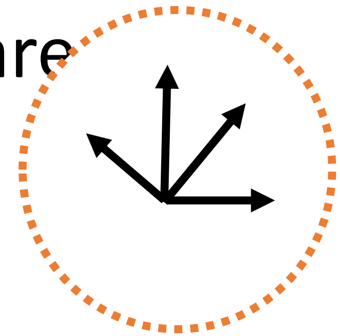

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Then there is a partition $T_1 \cup T_2$ such that **each part** has energy close to half in each direction:

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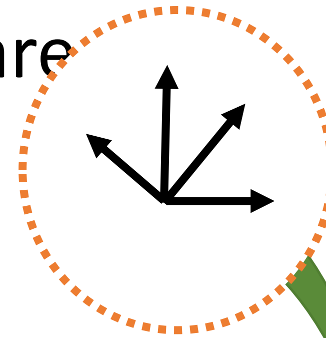

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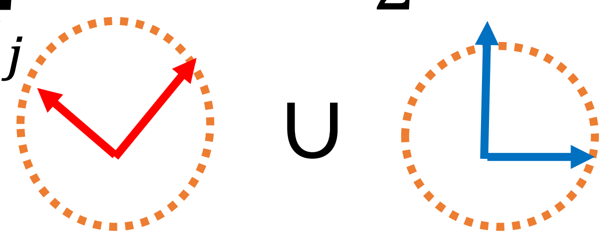

“Each part approximates the whole.”

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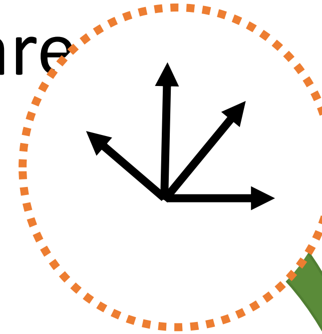

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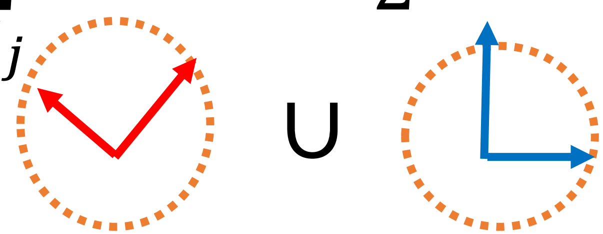
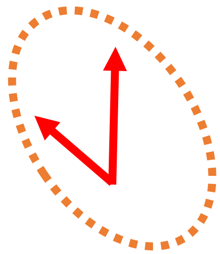
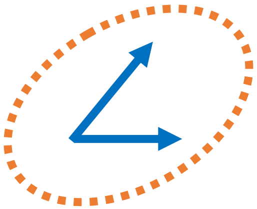
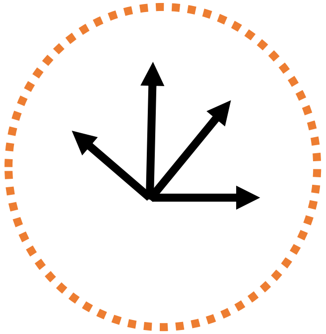


# Many Possible Partitions







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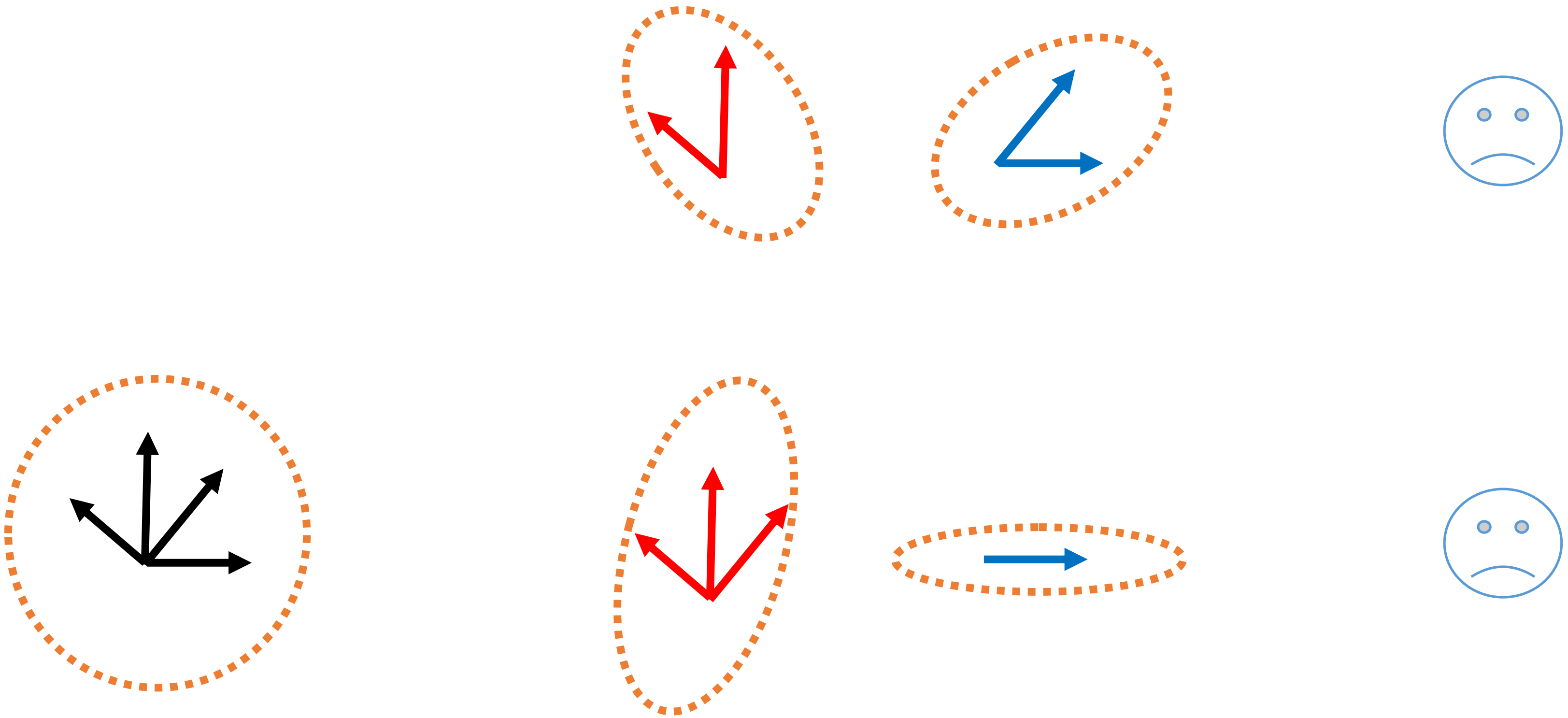

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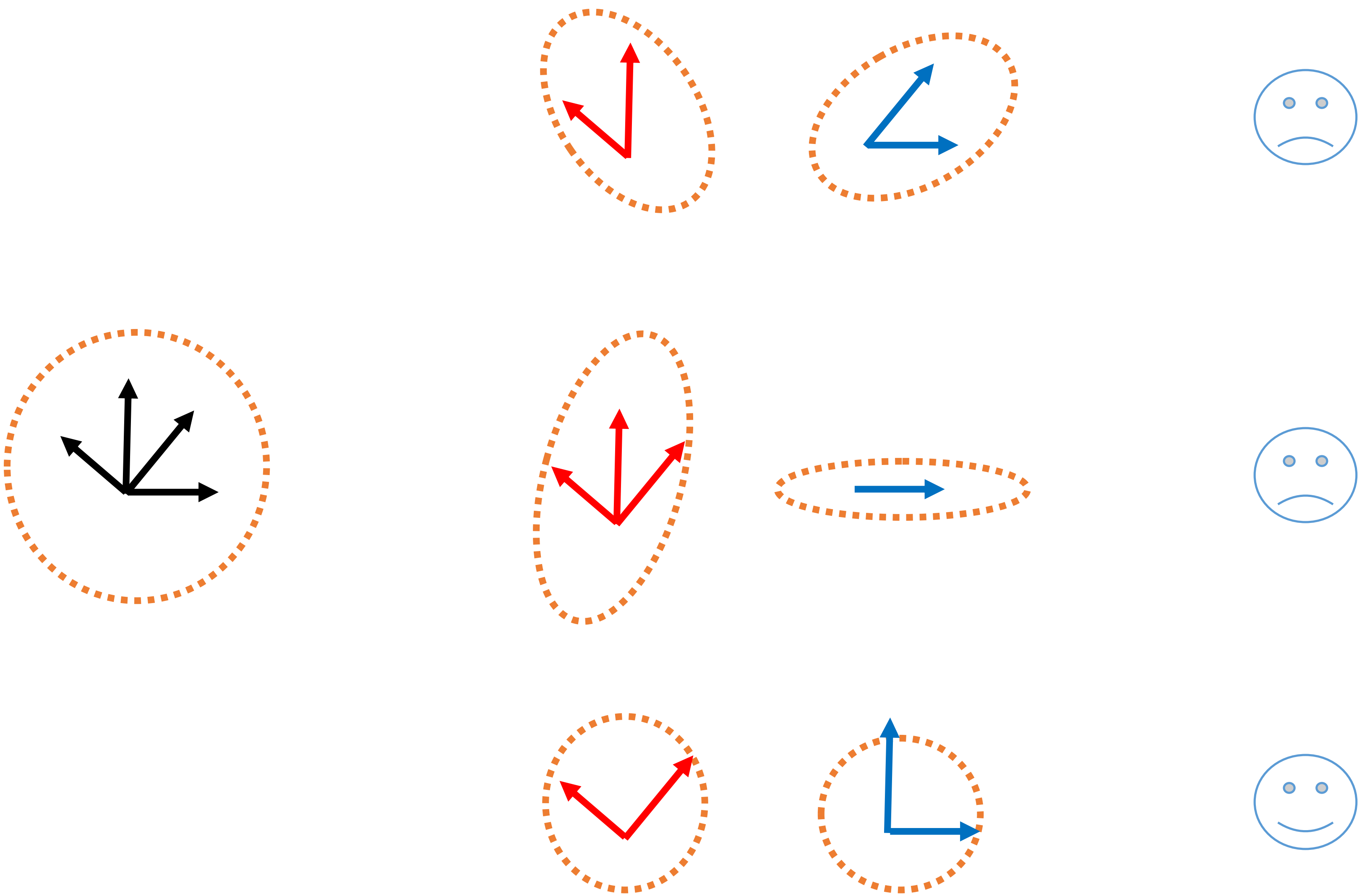

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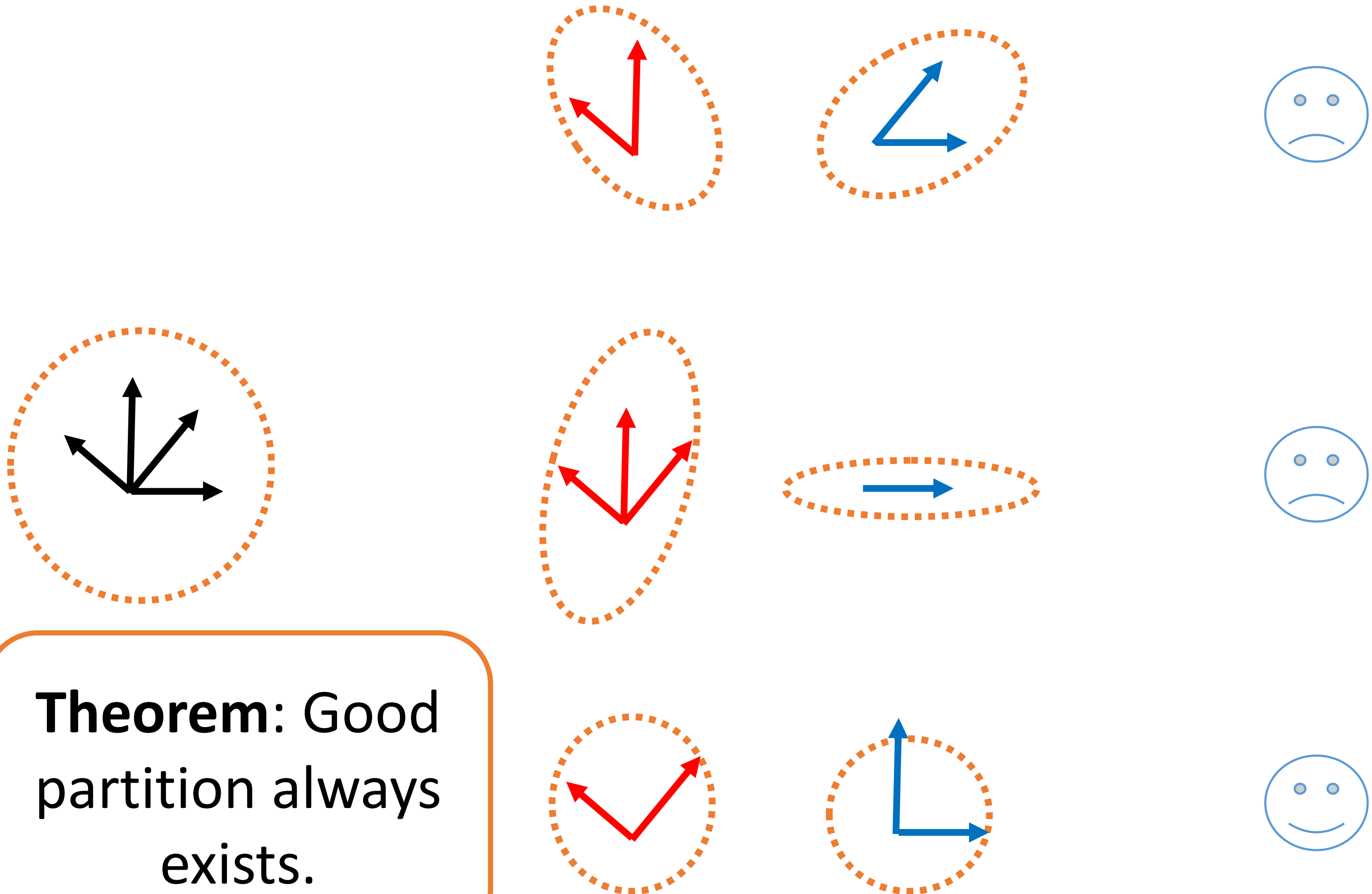

**Theorem**: Good partition always exists.

# The norm condition

**Main Theorem**. Suppose $v_1, \ldots, v_m \in \boldsymbol{R}^n$ are vectors $||v_i|| \leq \epsilon$ and energy one in each direction:

$$\forall ||u|| = 1 \qquad \sum_i \langle u, v_i \rangle^2 = 1$$

Then there is a partition $T_1 \cup T_2$ such that **each part** has energy close to half in each direction:

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**Main Theorem**. Suppose $v_1, \ldots, v_m \in \boldsymbol{R}^1$ are numbers with $|v_i| \leq \epsilon$ and energy one

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greedy bin packing

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Optimal in high dim

# Matrix Notation

Given vectors $v_1, \ldots, v_m$ write quadratic form as

$$Q(u) = \sum_i \langle v_i, u \rangle^2 = u^T \left( \sum_i v_i v_i^T \right) u$$

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**Comparision:**

$$A \preccurlyeq B \iff u^T A u \leq u^T B\, u \qquad \forall u$$

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Then there is a partition $T_1 \cup T_2$ such that

$$\boxed{\left(\frac{1}{2} - 5\epsilon\right) I \preccurlyeq \sum_{i \in T_j} v_i v_i^T \preccurlyeq \left(\frac{1}{2} + 5\epsilon\right) I}$$

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# Unnormalized Version

Suppose I get some vectors $w_1, \ldots, w_m$ which are **not** isotropic:

$$\sum_i w_i w_i^T = W \succcurlyeq 0$$

Consider $v_i \coloneqq W^{-\frac{1}{2}} w_i$ and apply theorem to $v_i$.

Normalized vectors have $||v_i||^2 = ||W^{-\frac{1}{2}} w_i||^2 = \epsilon$

Thm. gives

$$\left(\frac{1}{2} - 5\sqrt{\epsilon}\right) I \preccurlyeq \sum_{i \in T_j} v_i v_i^T \preccurlyeq \left(\frac{1}{2} + 5\sqrt{\epsilon}\right) I$$

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$$\left(\frac{1}{2} - 5\sqrt{\epsilon}\right) I \preccurlyeq W^{-\frac{1}{2}} \left( \sum_{i \in T_j} w_i w_i^T \right) W^{-\frac{1}{2}} \preccurlyeq \left(\frac{1}{2} + 5\sqrt{\epsilon}\right) I$$

# Unnormalized Version

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**Fact:** $A \preccurlyeq B \iff CAC \preccurlyeq CBC$
for invertible $C$

$\ldots_i w_i^{\ldots} - M \succcurlyeq 0$

Consider $v_i := W\ldots$ and apply theorem to $v_i$.

Normalized vectors have $||v_i||^2 = ||W^{-\frac{1}{2}} w_i||^2 = \epsilon$

$$\boxed{\left(\frac{1}{2} - 5\sqrt{\epsilon}\right) I \preccurlyeq W^{-\frac{1}{2}} \left( \sum_{i \in T_j} w_i w_i^T \right) W^{-\frac{1}{2}} \preccurlyeq \left(\frac{1}{2} + 5\sqrt{\epsilon}\right) I}$$

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$$\sum_i w_i w_i^T = W \succcurlyeq 0$$

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$$\left(\frac{1}{2} - 5\sqrt{\epsilon}\right) W \preccurlyeq \left(\sum_{i \in T_j} w_i w_i^T\right) \preccurlyeq \left(\frac{1}{2} + 5\sqrt{\epsilon}\right) W$$

# Unnormalized Theorem

Given arbitrary vectors $w_1, \ldots, w_m \in \mathbb{R}^n$ there is a partition $[m] = T_1 \cup T_2$ with

$$\left(\frac{1}{2} - \sqrt{\epsilon}\right)\left(\sum_i w_i w_i^T\right) \preccurlyeq \sum_{i \in T_j} w_i w_i^T \preccurlyeq \left(\frac{1}{2} - \sqrt{\epsilon}\right)\left(\sum_i w_i w_i^T\right)$$

Where $\epsilon := \max_i || W^{-\frac{1}{2}} w_i ||^2$

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Where $\epsilon := \max_i || W^{-\frac{1}{2}} w_i ||^2$

Any quadratic form in which no vector has too much influence can be split into two representative pieces.

# Applications

# 1. Graph Theory

Given an undirected graph $G = (V, E)$, consider its Laplacian matrix:

$$L_G = \sum_{ij \in E} \left(\delta_i - \delta_j\right)\left(\delta_i - \delta_j\right)^T$$

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$$\begin{array}{cc} \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ & \begin{array}{cc} i & \quad j \end{array} \end{array}$$

# 1. Graph Theory

Given an undirected graph $G = (V, E)$, consider its Laplacian matrix:

$$L_G = \sum_{ij \in E} (\delta_i - \delta_j)(\delta_i - \delta_j)^T = D - A$$

# 1. Graph Theory

Given an undirected graph $G = (V, E)$, consider its Laplacian matrix:

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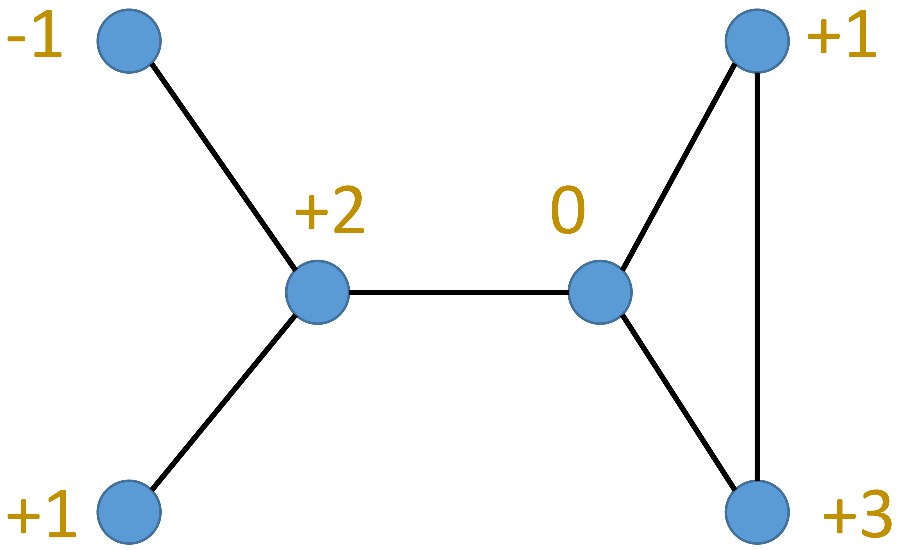
Quadratic form:

$$x^T L x = \sum_{ij \in E} (x_i - x_j)^2 \ for\ x \in \boldsymbol{R}^n$$

# The Laplacian Quadratic Form

An example:



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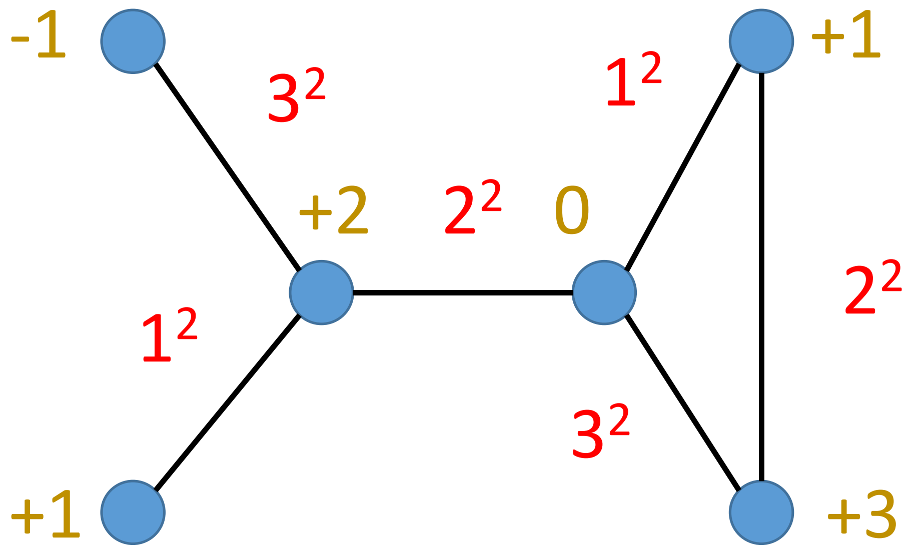


$$\boldsymbol{x}^T \boldsymbol{L} \boldsymbol{x} = \sum_{i,j \in E} (\mathbf{x}(i) - \mathbf{x}(j))^2$$

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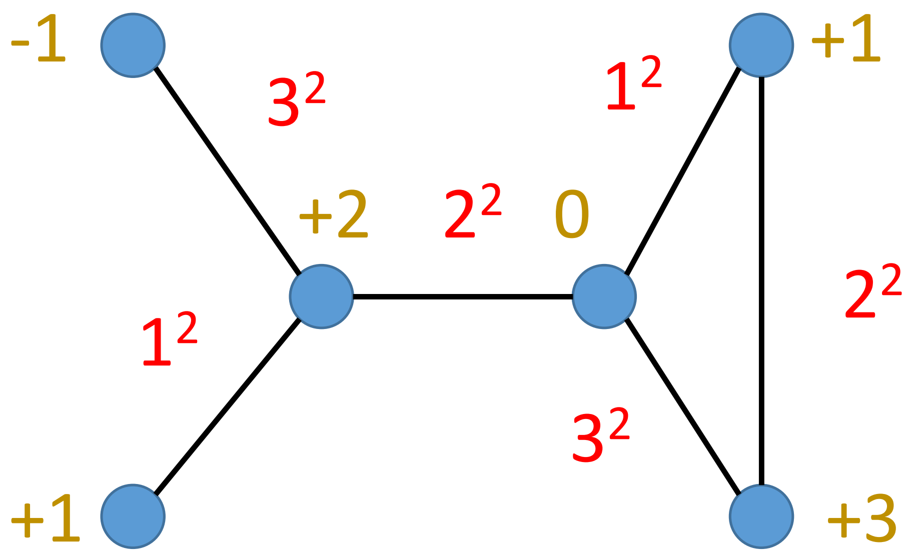
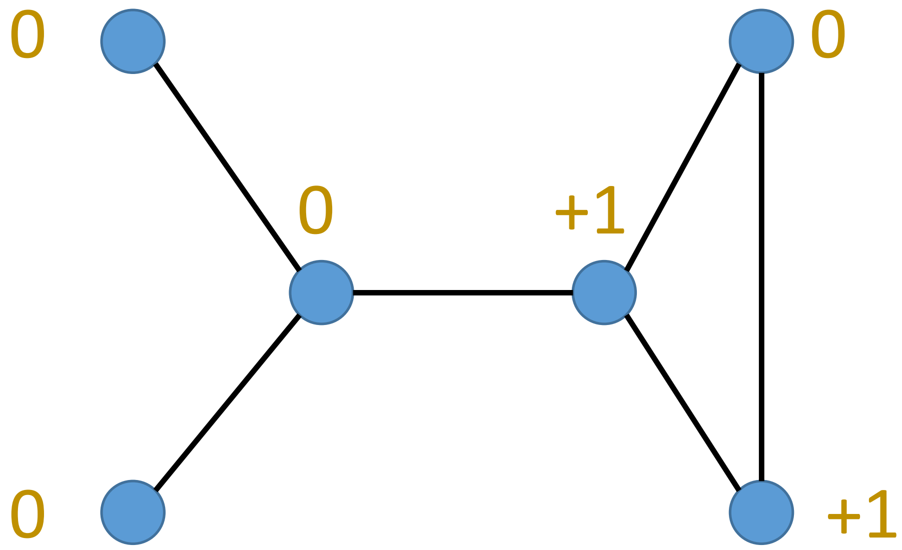



$$\boldsymbol{x}^T \boldsymbol{L x} = \sum_{i,j \in E} (\mathbf{x}(i)-\mathbf{x}(j))^2 = 28$$

# The Laplacian Quadratic Form

Another example:



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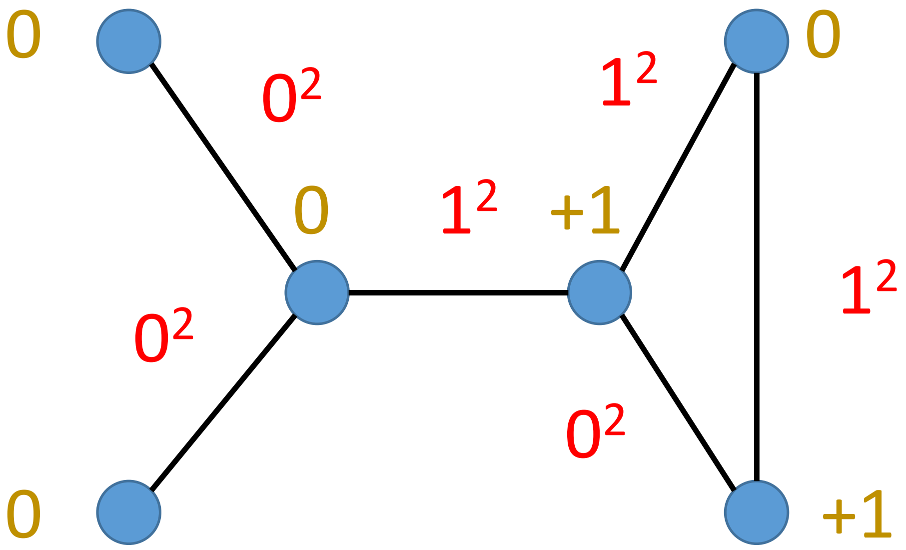


$$x^T L_G x = 3$$

# Cuts and the Quadratic Form

For characteristic vector $x_S \in \{0,1\}^n$ of $S \subseteq V$

$$
\begin{aligned}
x_S^T L_G x_S &= \sum_{ij \in E} w_{ij}(x(i) - x(j))^2 \\
&= \sum_{ij \in (S,\overline{S})} w_{ij} \\
&= wt_G(S, \overline{S})
\end{aligned}
$$

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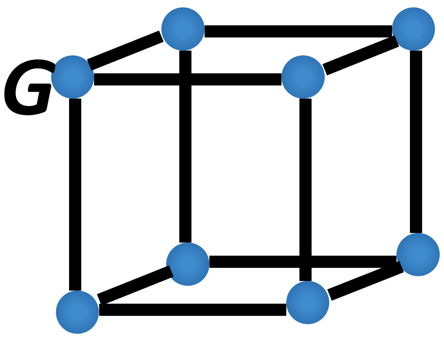
The Laplacian Quadratic form encodes the entire cut structure of the graph.

# Application to Graphs



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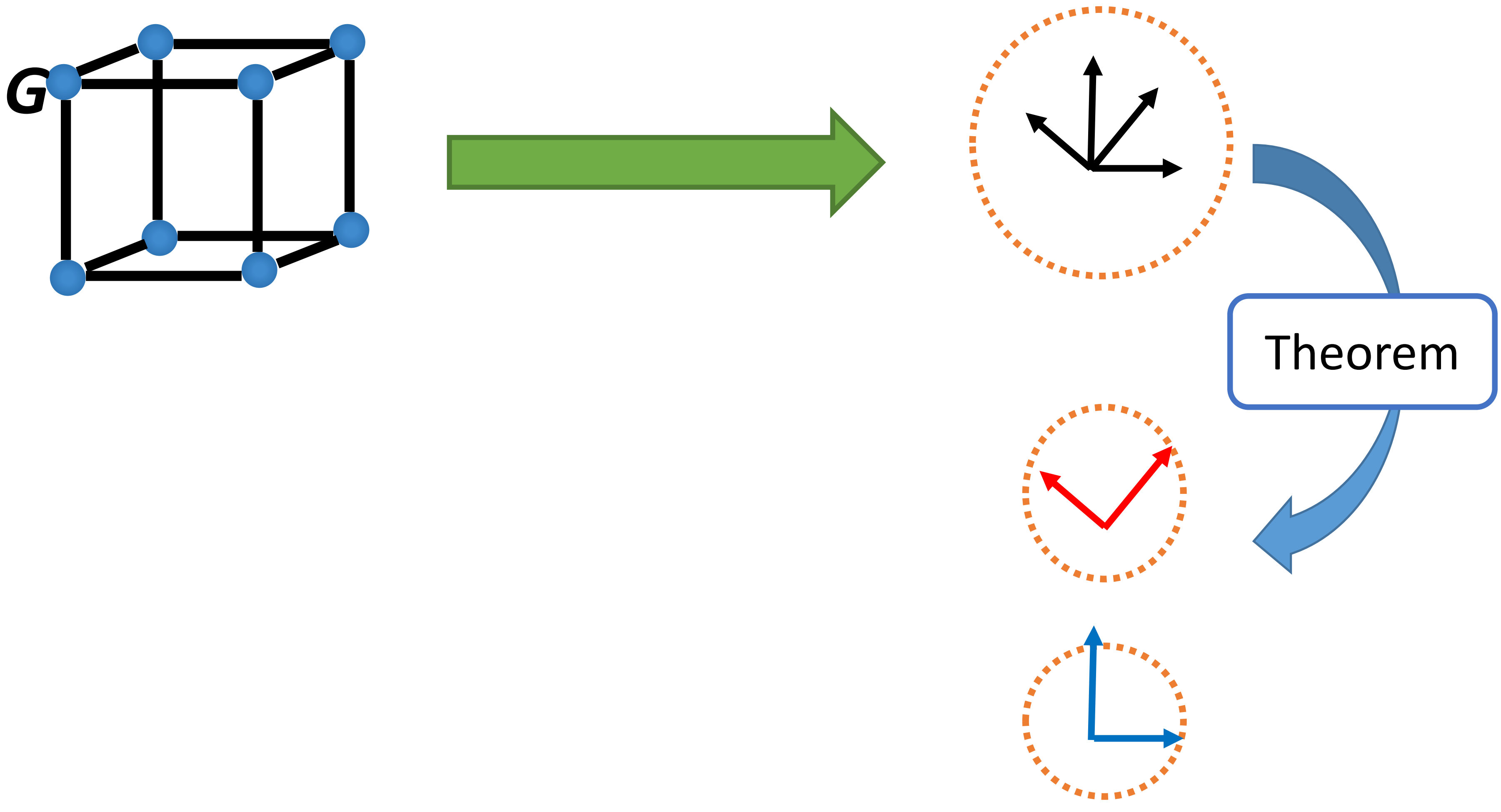

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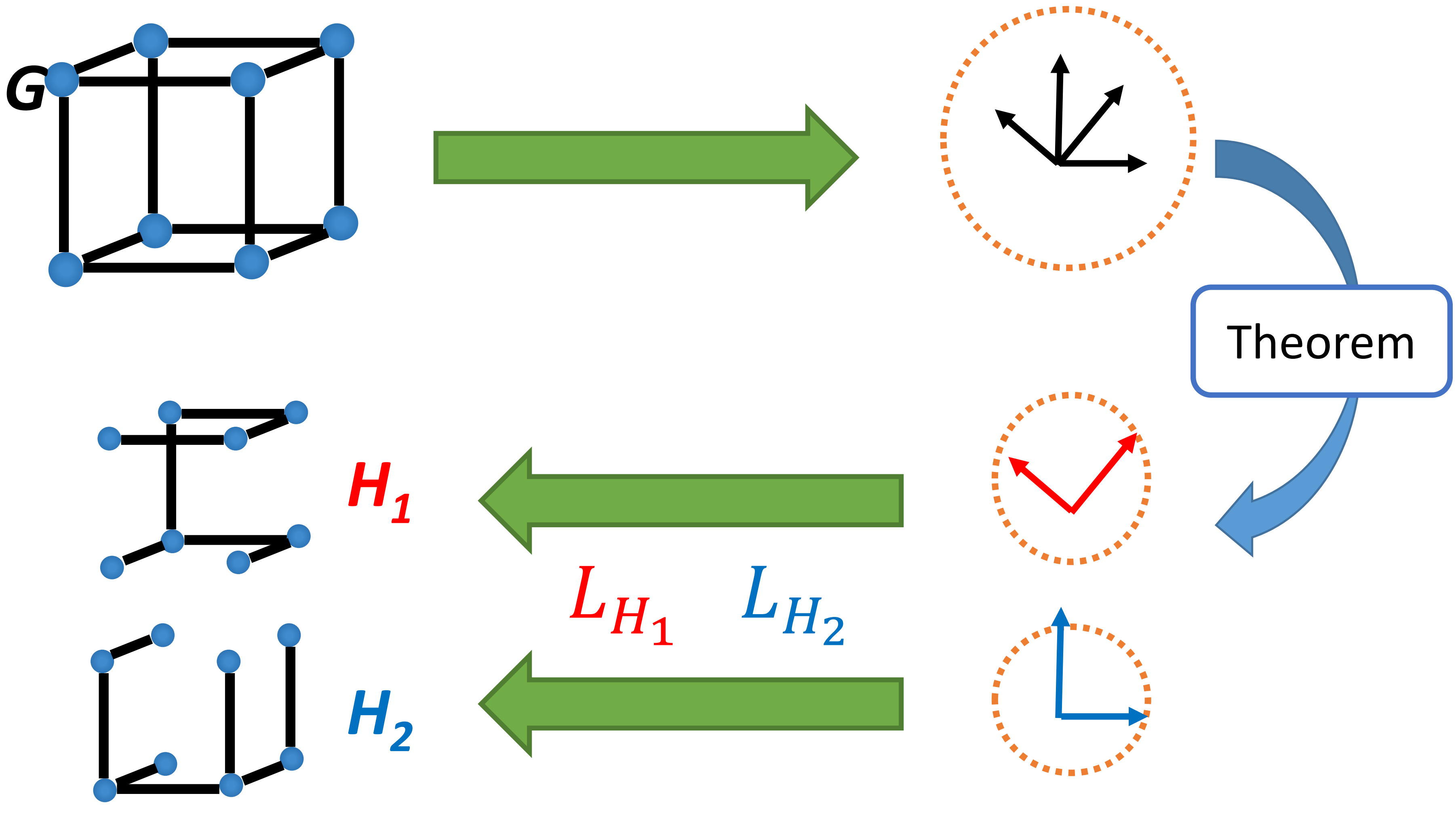

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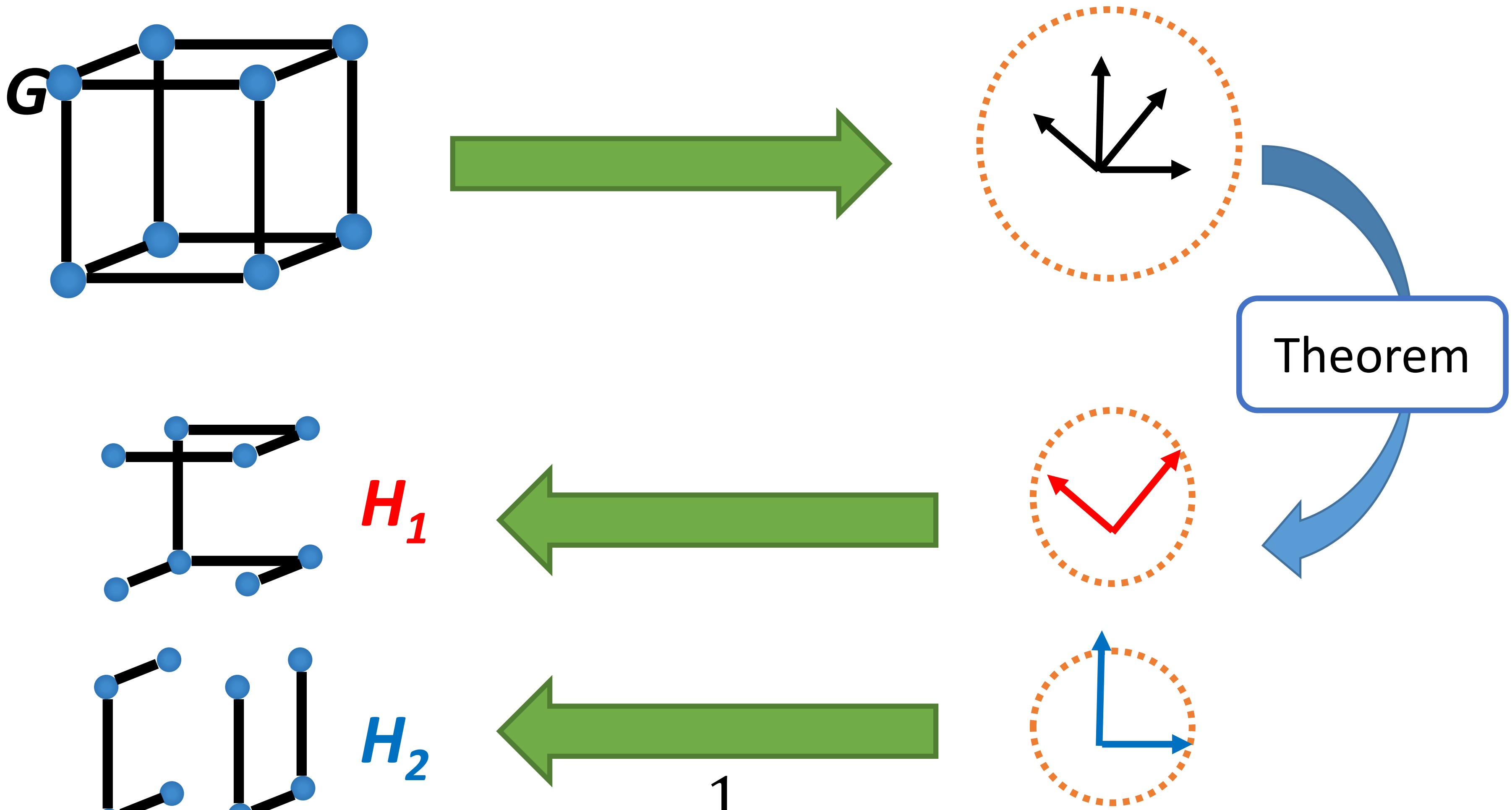
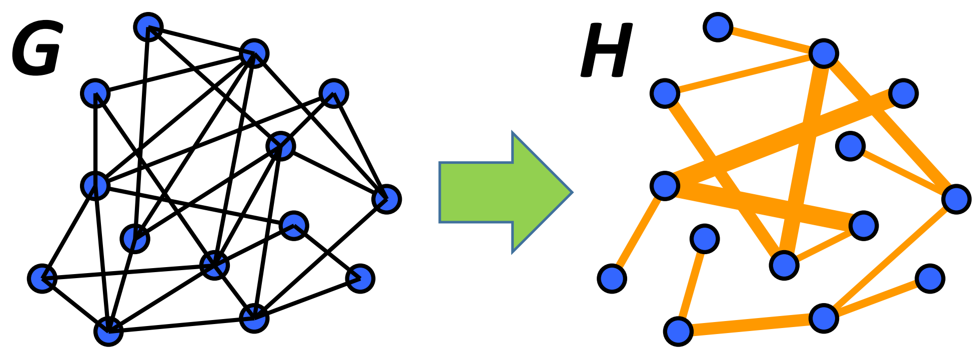



$$x^T L_{H_1} x \approx \frac{1}{2} x^T L_G x$$

# Recursive Application Gives:

**1. Graph Sparsification Theorem [Batson-Spielman-S'09]:** Every graph **G** has a *weighted* O(1)-cut approximation **H** with O(n) edges.

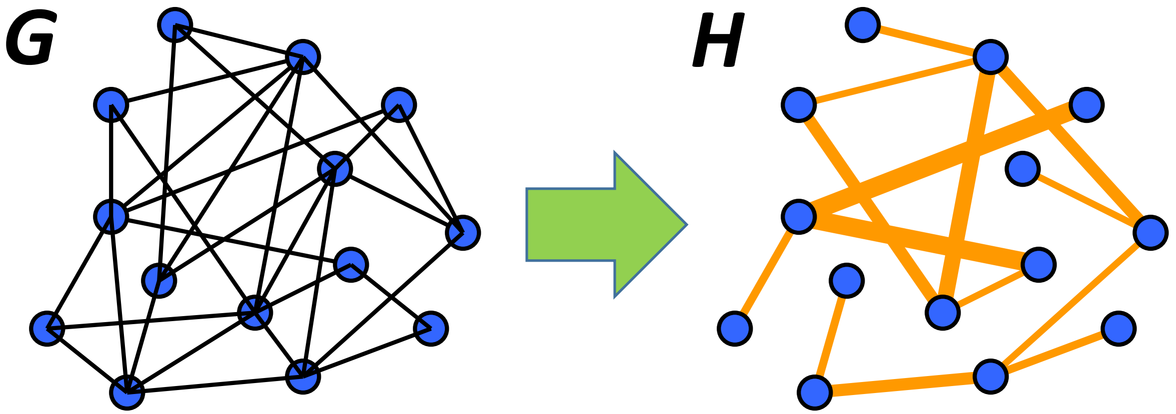



$O(n^2)$ edges $\qquad$ $O(n)$ edges

Unweighted $\qquad$ **Weighted**

# Approximating One Graph by Another

**Cut Approximation** [Benczur-Karger'96]




For **every** cut,

weight of edges in **G** ≈ weight of edges in **H**

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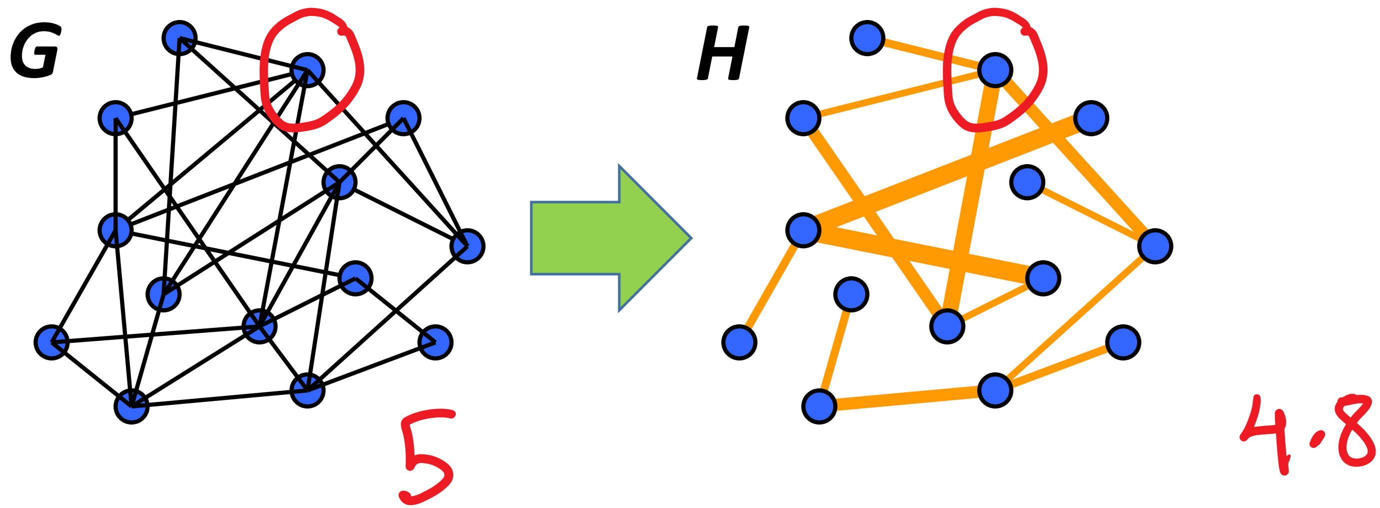


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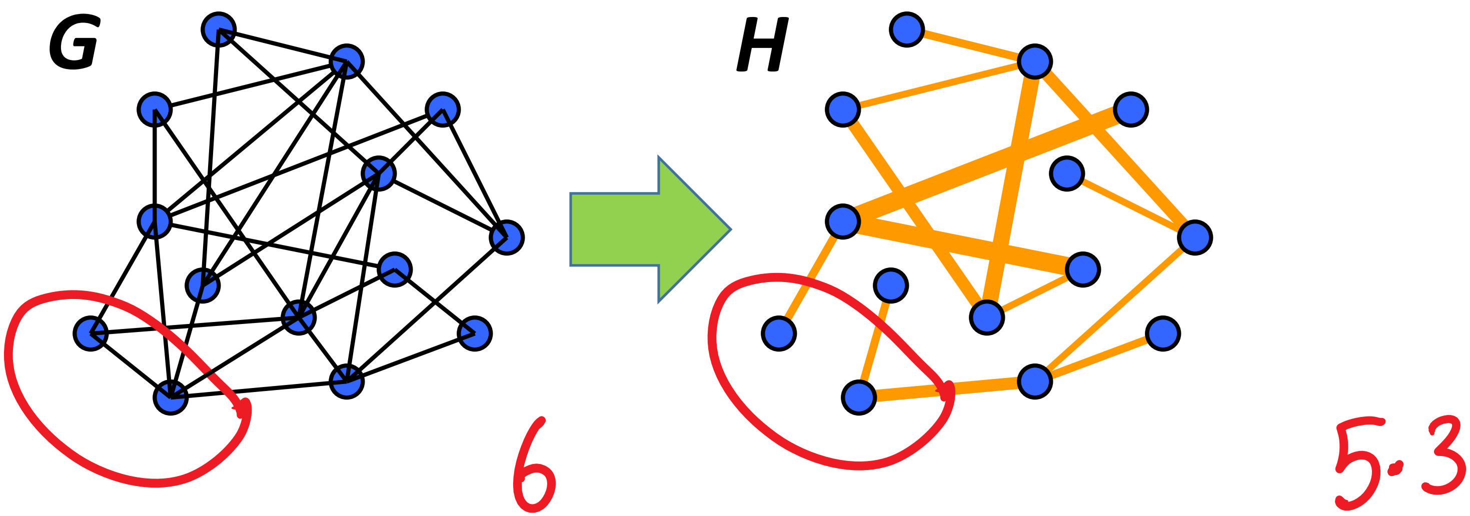


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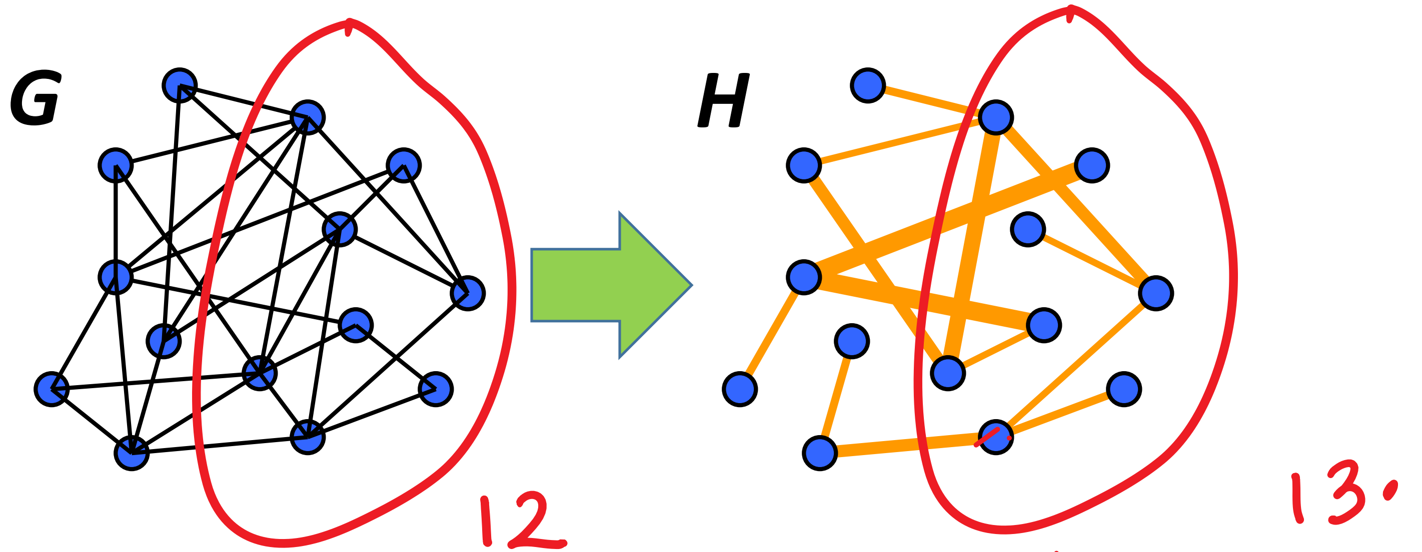


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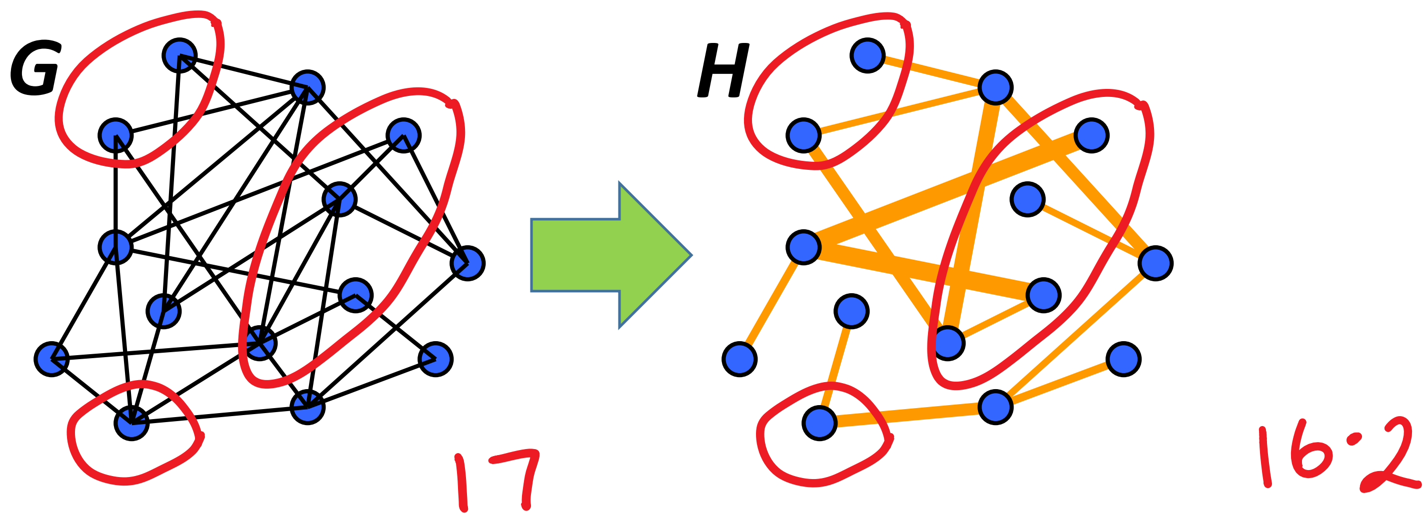


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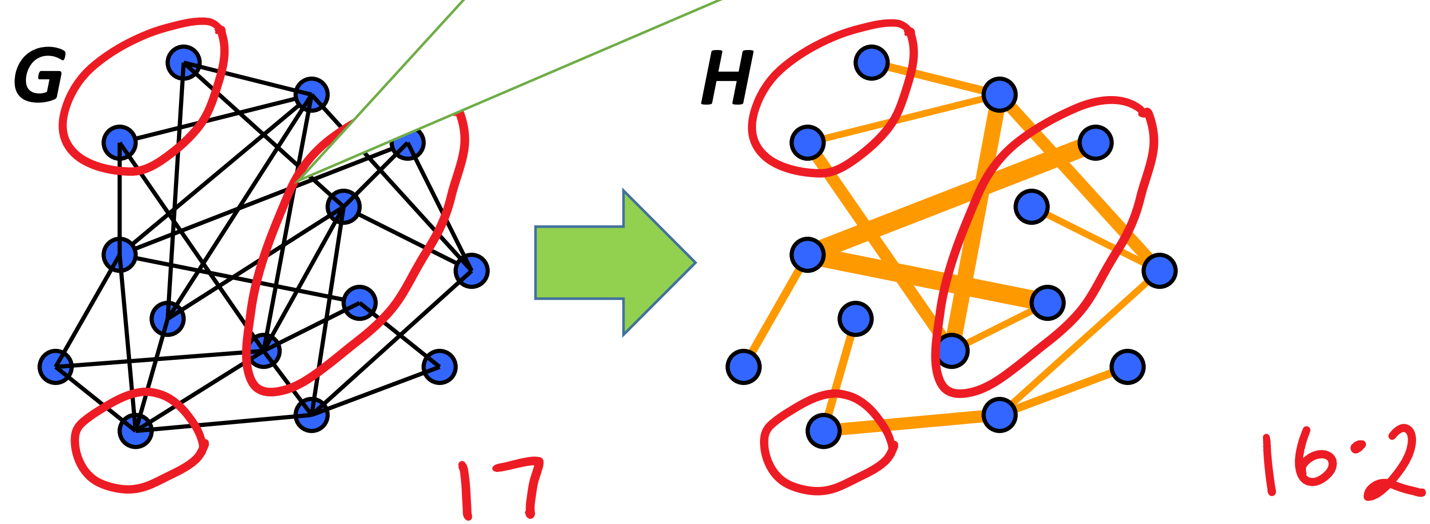
# Approximating One Graph by Another

Cut Approximati

*G* and *H* have same cuts. Equivalent for min cut, max cut, sparsest cut…




For **every** cut,

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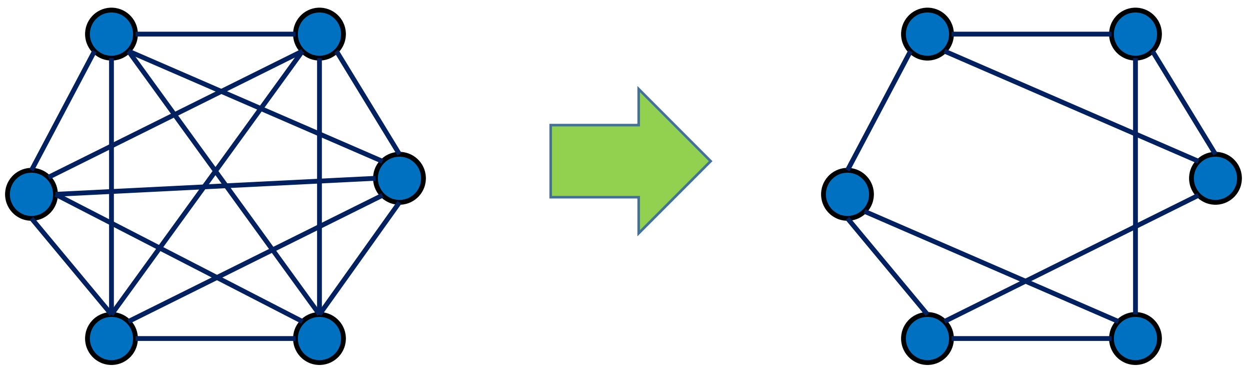
**2. Unweighted Graph Sparsification** Every transitive graph **G** has an unweighted O(1)-cut approximation **H** with O(n) edges.



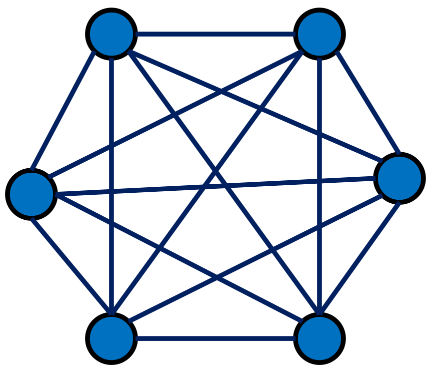
$K_n$

$H$

Expander graph

# Recursive Application Gives:

**2. Unweighted Graph Sparsification** Every transitive graph **G *can be partitioned into*** O(1)-cut approximations with O(n) edges.



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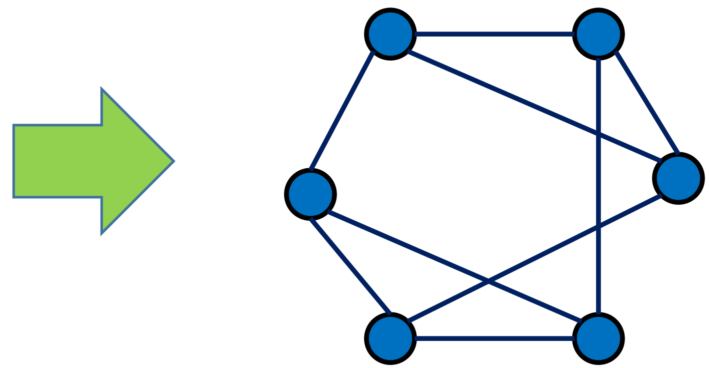

$H_1 \ldots H_n$

Expander graphs

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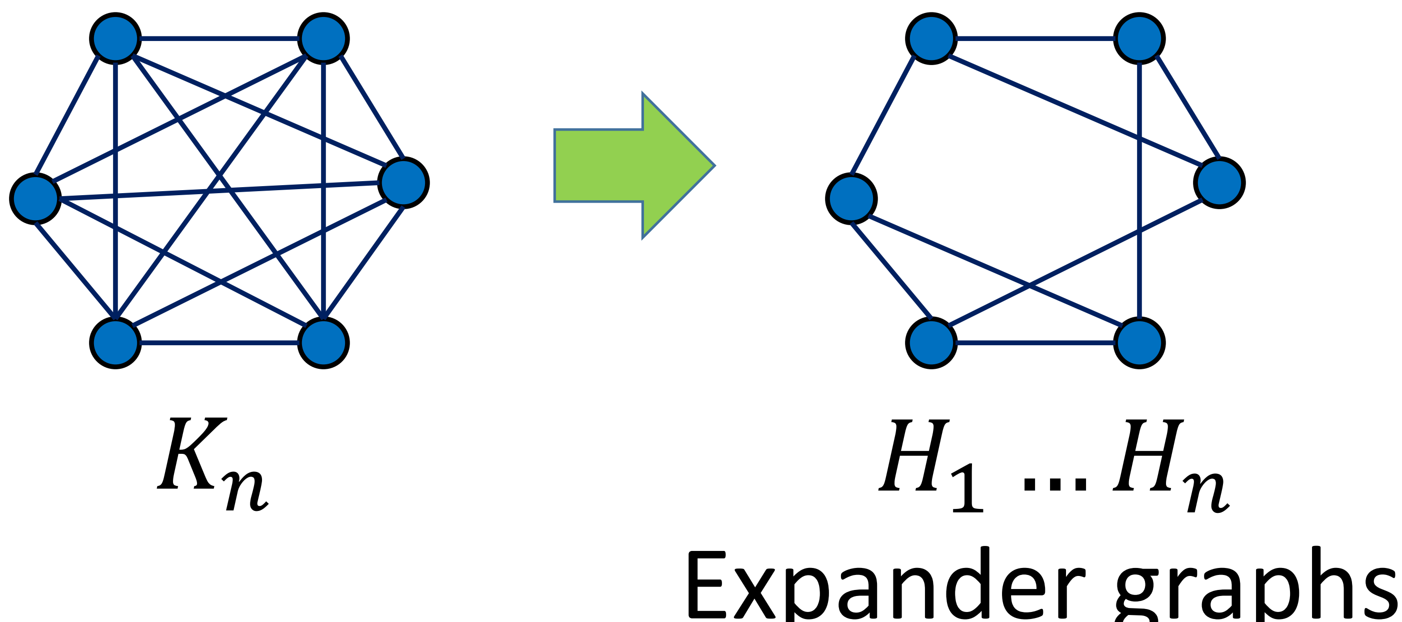


Generalizes [Frieze-Molloy]

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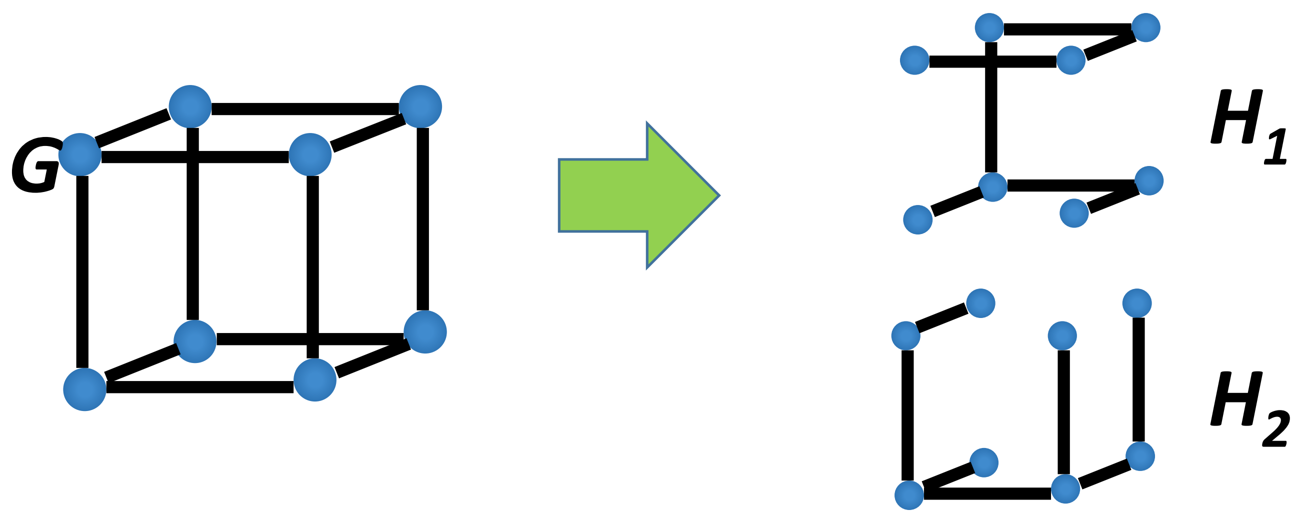


Same cut structure

□

# 2. Uncertainty Principles

Signal $x \in \mathbb{C}^n$. Discrete Fourier Transform

$$\hat{x}(\mathrm{a}) = \langle x, \left(\exp(-\frac{a2\pi ik}{n})\right)_{k\leq n}\rangle$$

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If $x$ is supported on $|S| = \sqrt{n}$ coordinates,

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**Stronger Uncertainty Principle:**

For every subset $|S| = \sqrt{n}$, there is a partition

$$[n] = T_1 \cup \cdots T_{\sqrt{n}}$$

$$||x|_S||_2 \approx \frac{1}{\sqrt{n}} ||\hat{x}|_{T_i}||_2$$ for all $x$ and $T_i$

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# 2. Uncertainty Principles

**Proof.**

Let $f_k = \left(\exp(-\frac{a2\pi ik}{n})\right)_{k\leq n}$ be the Fourier basis.

Fix a subset $S \subset [n]$ of $\sqrt{n}$ coords.

The restricted norm is:

$$||x|_S||^2 = \sum_k \langle x|_S, f_k\rangle^2$$

a quadratic form in $\sqrt{n}$ dimensions.

**Apply the theorem.**

$\square$

# 2. Uncertainty Principles

Applications in analytic number theory, harmonic analysis.

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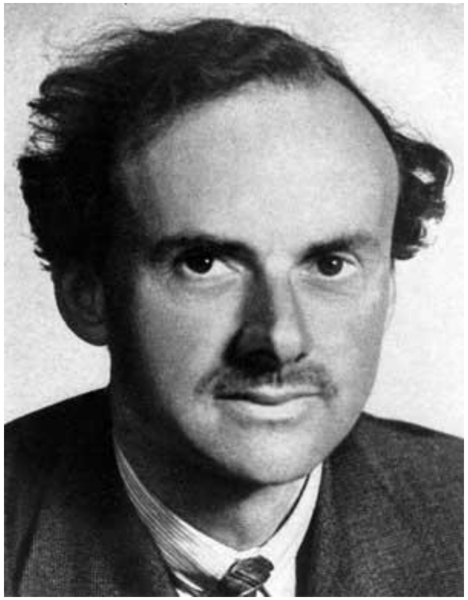
# 3. The Kadison-Singer Problem

**Dirac 1930's:** Bra-Ket formalism for quantum states.

$$|\psi\rangle, |p\rangle \ldots$$



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**Conjecture**: about $\infty$ matrices

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**Conjecture**: about ∞ matrices

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

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**Kadison-Singer 1959:** Does this lead to a

**Anderson 1979**: Reduced to a question about finite matrices. “Paving Conjecture”

$$\begin{bmatrix} a_{11} & \cdots & \cdots \\ \vdots & & \\ \vdots & & \end{bmatrix} \in \mathbb{C}^{n \times n}$$

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**Weaver 2002:** Discrepancy theoretic formulation of the same question.

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**This work:** Proof of Weaver’s conjecture.

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$\square$

# In General

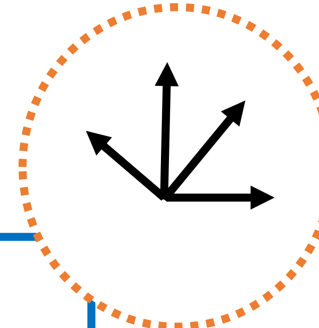
*Anything* that can be encoded as a quadratic form can be split into pieces while preserving certain properties.

Many different things can be gainfully encoded this way.

# Proof

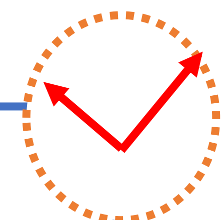
# Main Theorem

Suppose $v_1, \ldots, v_m \in \boldsymbol{R}^n$ are vectors $||v_i||^2 \leq \epsilon$ and

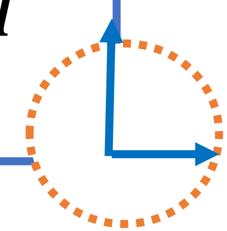


$$\sum_i v_i v_i^T = I_n$$

Then there is a partition $T_1 \cup T_2$ such that



$$\left(\frac{1}{2} - 5\sqrt{\epsilon}\right) I \preccurlyeq \sum_{i \in T_j} v_i v_i^T \preccurlyeq \left(\frac{1}{2} + 5\sqrt{\epsilon}\right) I$$



# Equivalent Theorem

Suppose $v_1, \dots, v_m \in \boldsymbol{R}^n$ are vectors $||v_i||^2 \le \epsilon$ and

$$\boxed{\sum_i v_i v_i^T = I_n}$$

Then there is a partition $T_1 \cup T_2$ such that

$$\boxed{\sum_{i \in T_j} v_i v_i^T \preccurlyeq \left(\frac{1}{2} + 5\sqrt{\epsilon}\right) I}$$

# Equivaler

$$\sum_{i \in T_1} v_i v_i^T = I - \sum_{i \in T_2} v_i v_i^T$$

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Choose $T_1 \cup T_2 = [m]$ randomly

Want:

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**Trick**: embed in blocks of $2n \times 2n$ matrix

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$$\left\| \begin{bmatrix} \sum_{i \in T_1} v_i v_i^T & 0 \\ 0 & \sum_{i \in T_2} v_i v_i^T \end{bmatrix} \right\| = \max_{\mathrm{j}} \left\| \sum_{i \in T_j} v_i v_i^T \right\|$$

# Idea 1: Random Partition

Define independent random $v_1, \ldots, v_m \in \mathbb{R}^{2n}$

$$v_i = \begin{cases} \begin{pmatrix} v_i \\ 0 \end{pmatrix} & with\ prob\ 0.5 \\ \begin{pmatrix} 0 \\ v_i \end{pmatrix} & with\ prob.\ 0.5 \end{cases}$$

Then

$$\mathbb{E}\left\|\sum_i v_i v_i^T\right\| = \mathbb{E}_T \max_j \left\|\sum_{i \in T_j} v_i v_i^T\right\|$$

# The Matrix Chernoff Bound

$v_1, \ldots, v_m \in \mathbb{R}^{2n}$ are **independent**,

$\mathbb{E} \sum_i v_i v_i^T = \frac{I}{2}$ and $||v_i||^2 \leq \epsilon$

**Tropp 2011**

$$\mathbb{E} \left\| \sum_i v_i v_i^T \right\| \leq \frac{1}{2} + O(\sqrt{\epsilon \log n})$$

Analogous for the scalar Chernoff bound for sums of independent bdd random numbers.

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# Main Theorem

$\exists v_1, \ldots, v_m$ *such that*

$$||\sum_i v_i v_i^T|| \leq 1 + \sqrt{\epsilon}$$

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# Idea 2: The Expected Polynomial

Just like yesterday,

$$\left\|\sum_i v_i v_i^T\right\| = \lambda_{max}\left(\det(xI - \sum_i v_i v_i^T)\right)$$

Consider

$$\mu(x) := \mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$$

# 3-Step Plan

1. Show that there exist $v_1, \ldots, v_m$ with

$$\lambda_{max}\, \chi\left(\sum_i v_i v_i^T\right) \leq \lambda_{max}\, \mathbb{E}\chi\left(\sum_i v_i v_i^T\right)$$

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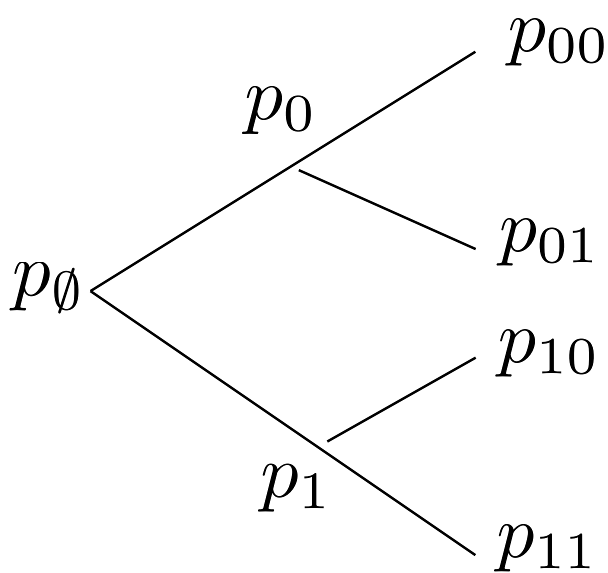
$$\boxed{\left\{\chi_{\left(\sum_i v_i v_i^T\right)}(x)\right\}_{v_i \sim v_i} \text{ is an interlacing family}}$$



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$\mathbb{E}\chi_{(\sum_i v_i v_i^T)}(x)$ is real-rooted for all product distributions on signings.

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$$\mathbb{E}\chi\left(\sum_i v_i v_i^T\right) = \prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right) \det\left(xI + \sum_i z_i A_i\right)\Bigg|_{z_1=\cdots=z_m=0}$$

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2. Calculate

$$\mathbb{E}\chi\left(\sum_i v_i v_i^T\right)$$

# Central Identity

Suppose $v_1, \ldots, v_m$ are **independent** random vectors with $A_i := \mathbb{E} v_i v_i^T$. Then

$$\mathbb{E} \det\left( xI - \sum_i v_i v_i^T \right)$$
$$= \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial z_i}\right) \det\left( xI + \sum_i z_i A_i \right) \Bigg|_{z_1 = \cdots = z_m = 0}$$

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Assuming $\mathbb{E} \sum_i v_i v_i^T = I$ and $||v_i||^2 \le \epsilon$

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# Bounding the Roots

Need to bound the roots of

$$\prod_{i=1}^{m}\left(1-\partial_{z_i}\right)\det\left(xI+\sum_i z_i A_i\right)\Bigg|_{z_1=\ldots z_m=0}$$

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**Answer:** Interlacing

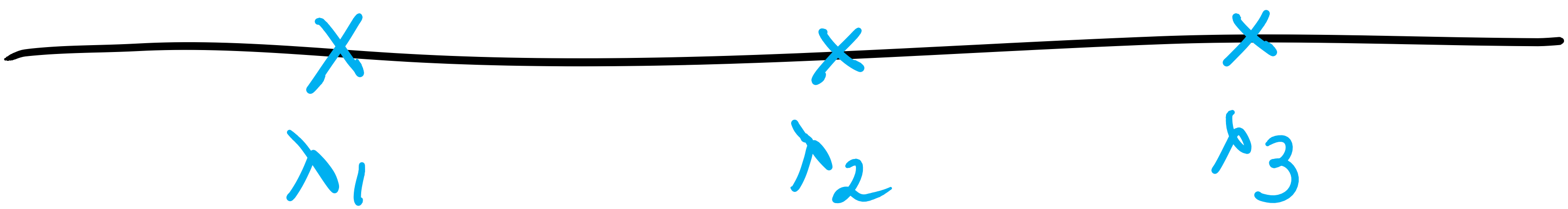
Consider $p(x) = (x-\lambda_1)\ldots(x-\lambda_n)$ distinct

$$p'(x) = \sum_j \prod_{i \neq j} (x-\lambda_i)$$

Then $\frac{p'(x)}{p(x)} = \sum_i \frac{1}{x-\lambda_i}$ is a rational function with

$n$ poles.

# A Rational Function

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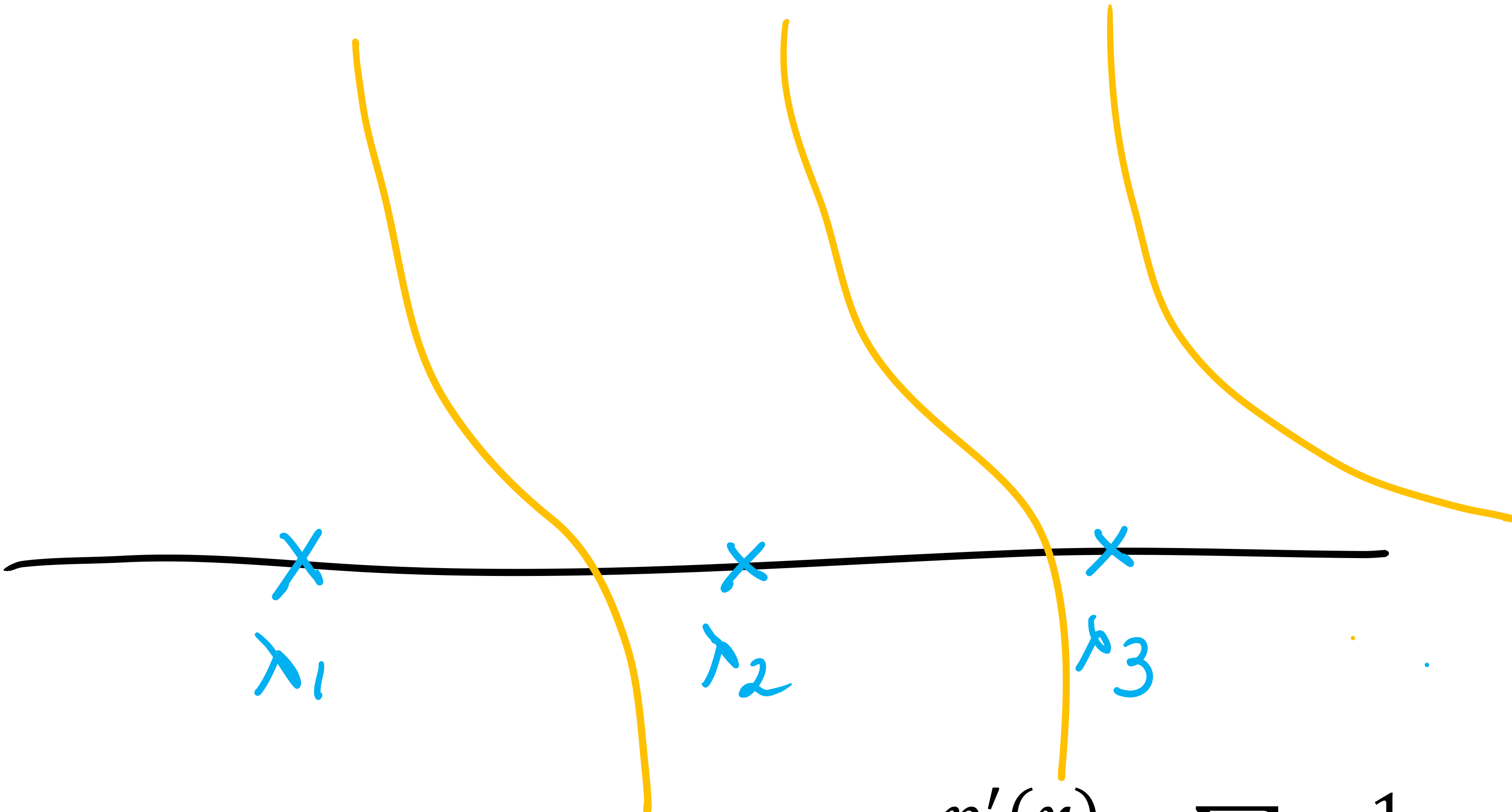


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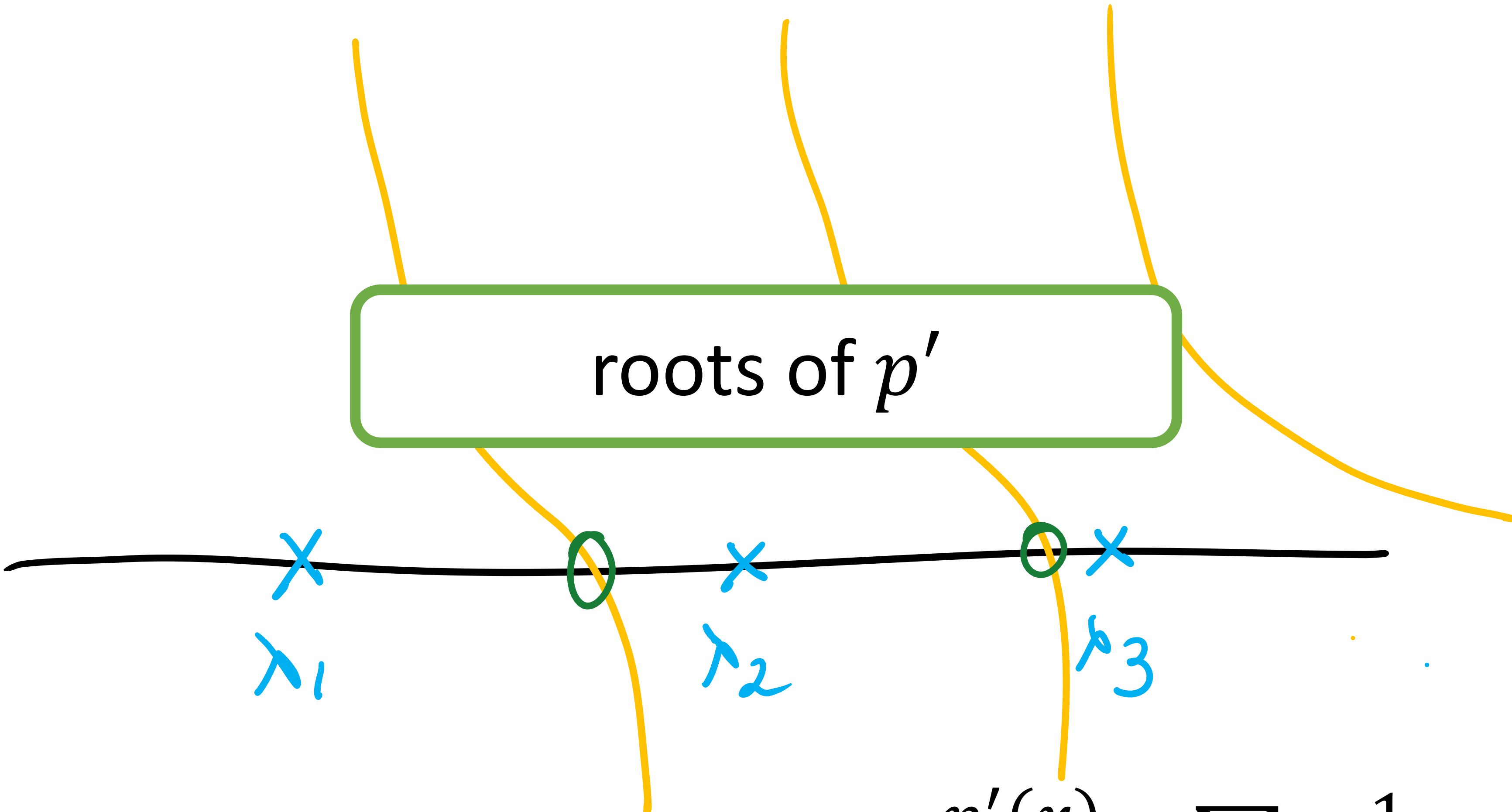


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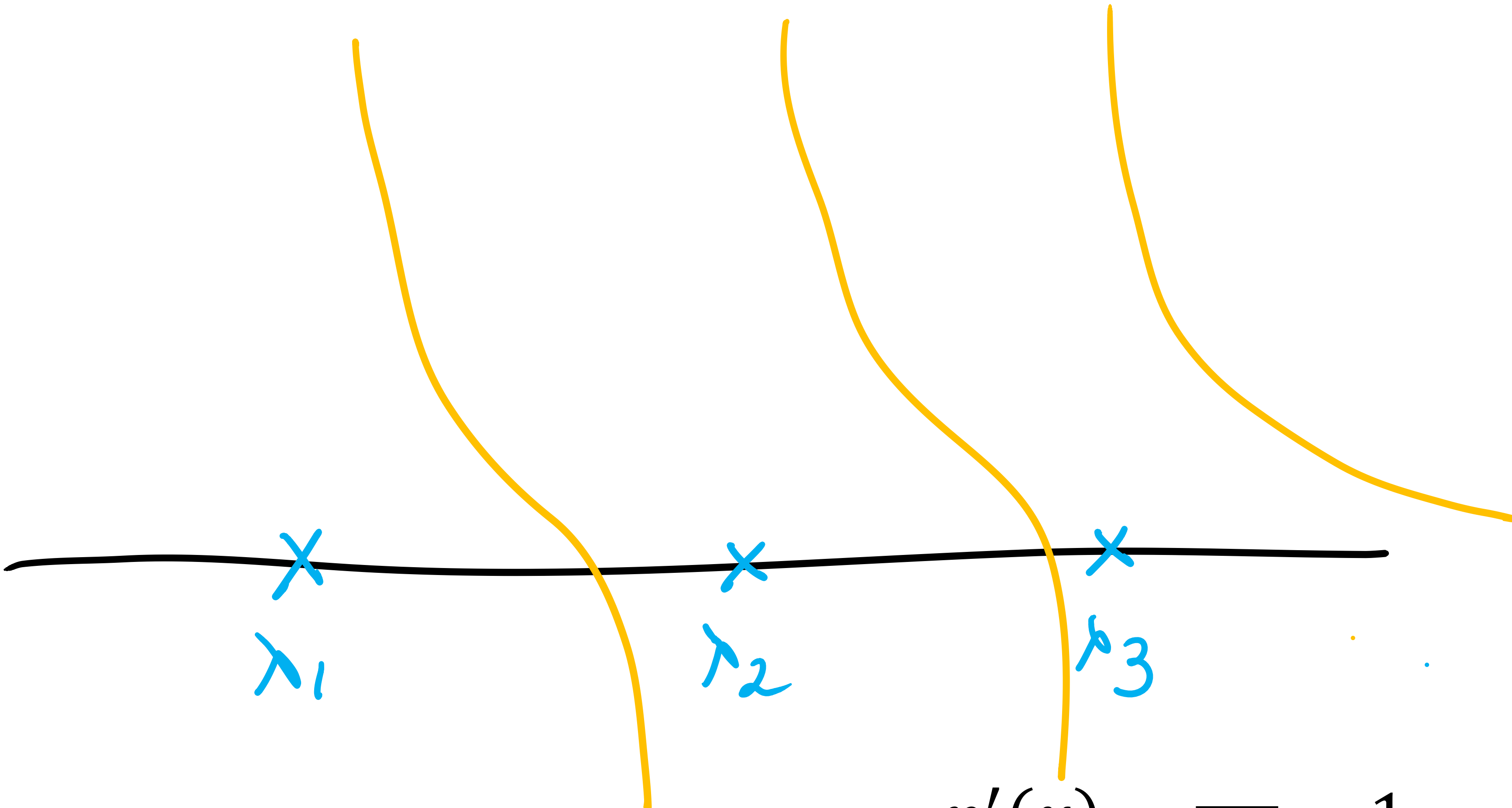


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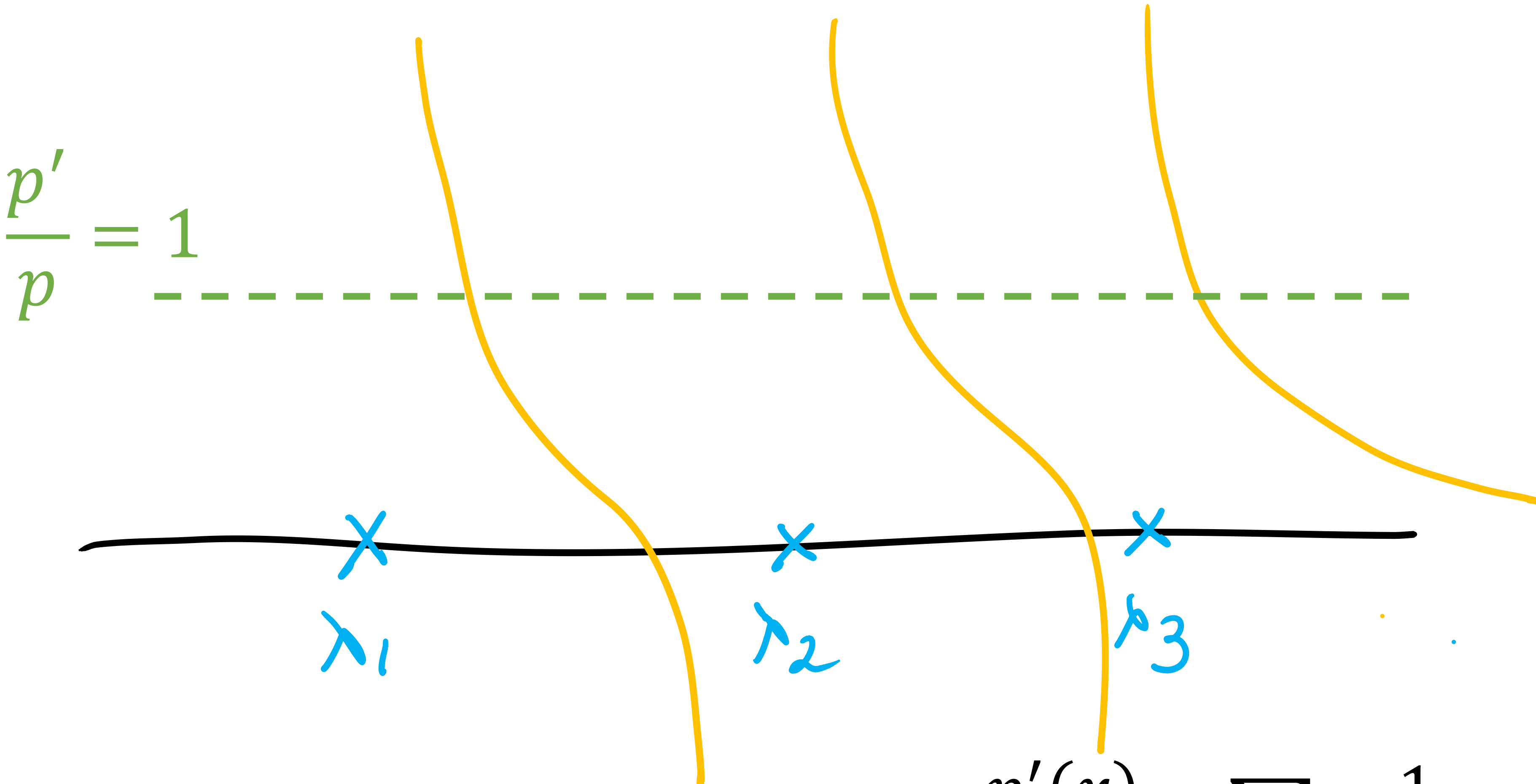


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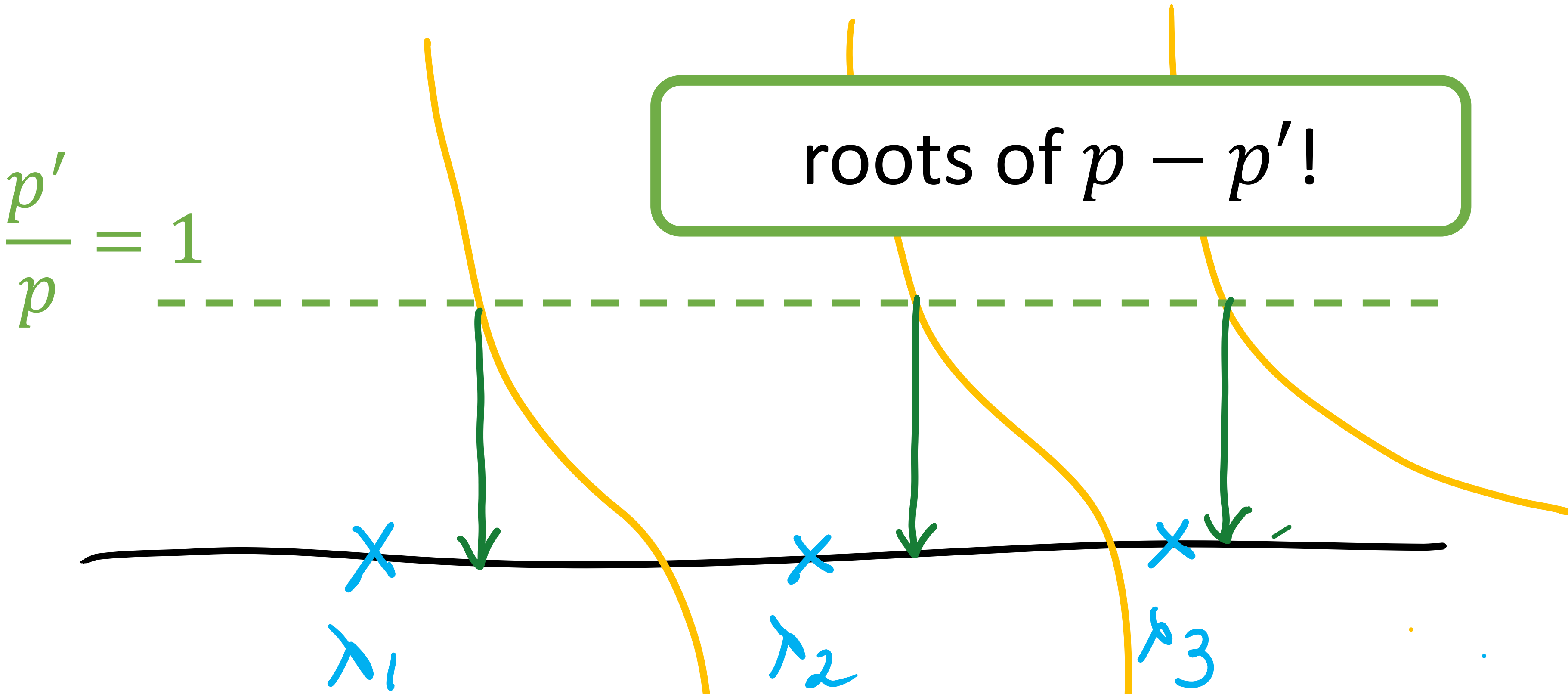


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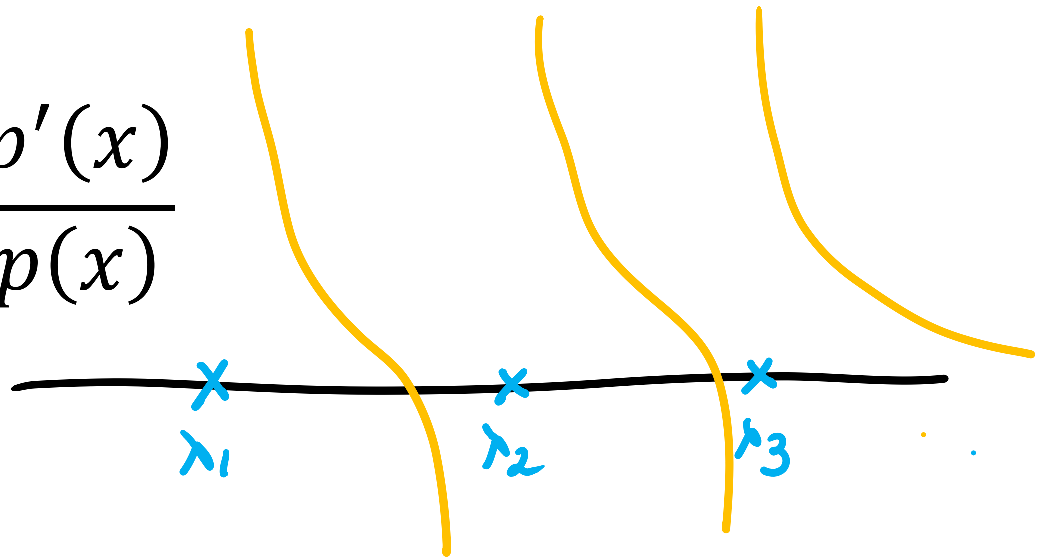
# The Barrier Function

**Define:**

$$\Phi_p(x) = \frac{p'(x)}{p(x)}$$



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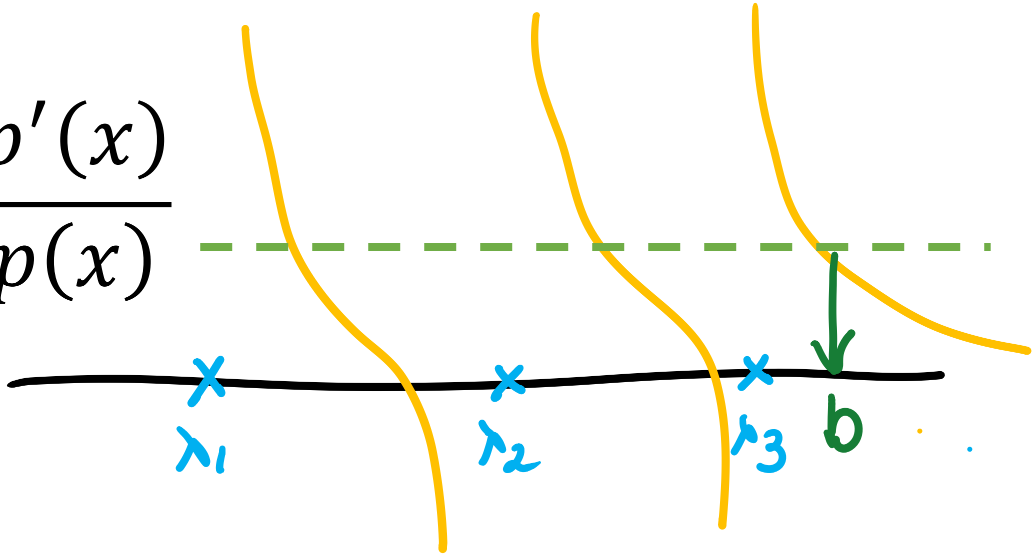


To bound roots of $p - p'$, find a point above the roots of $p$ with $\Phi_p(b) < 1$.

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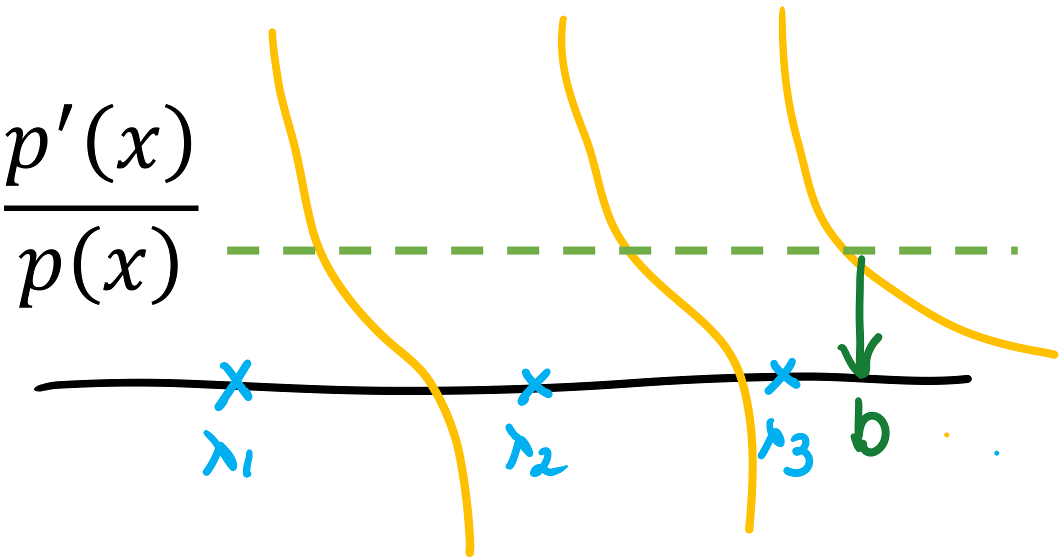


To bound roots of $p - p'$, find a point above the roots of $p$ with $\Phi_p(b) < 1$.

Level sets $\{\Phi_p < 1\}$ contain no zeros of $p - p'$

# The Barrier Method [BSS'09]

**Theorem.** If ***b*** is above the roots of ***p*** and

$\Phi_p(b) \leq 1 - 1/\delta$ then

$\Phi_{p-p'}(b+\delta) \leq 1 - 1/\delta$

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Relates level sets of $\Phi_p$ to level sets of $\Phi_{(1-\partial)p}$

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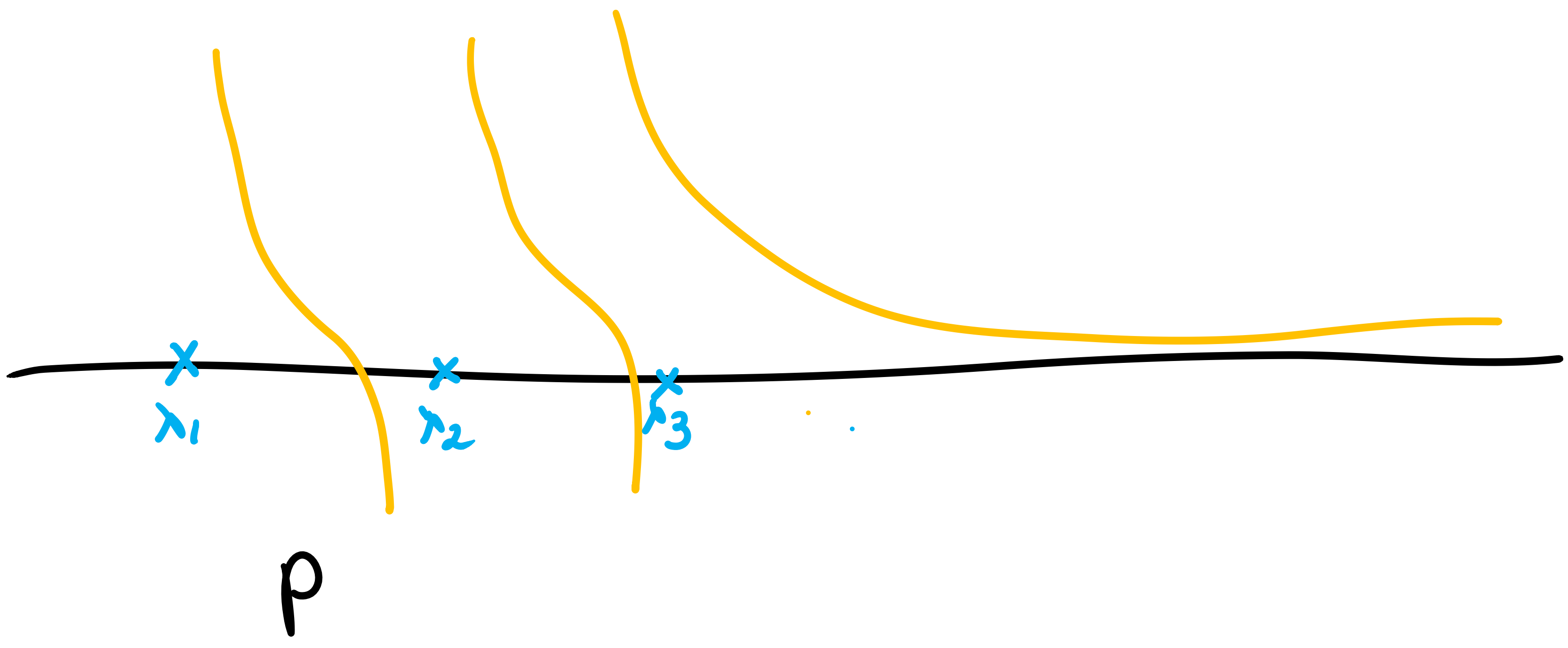
Relates level sets of $\Phi_p$ to level sets of $\Phi_{(1-\partial)p}$

Robust version of $\{\Phi_p < 1\}$ argument – can be iterated.

# Evolution of level sets of $\Phi_p$



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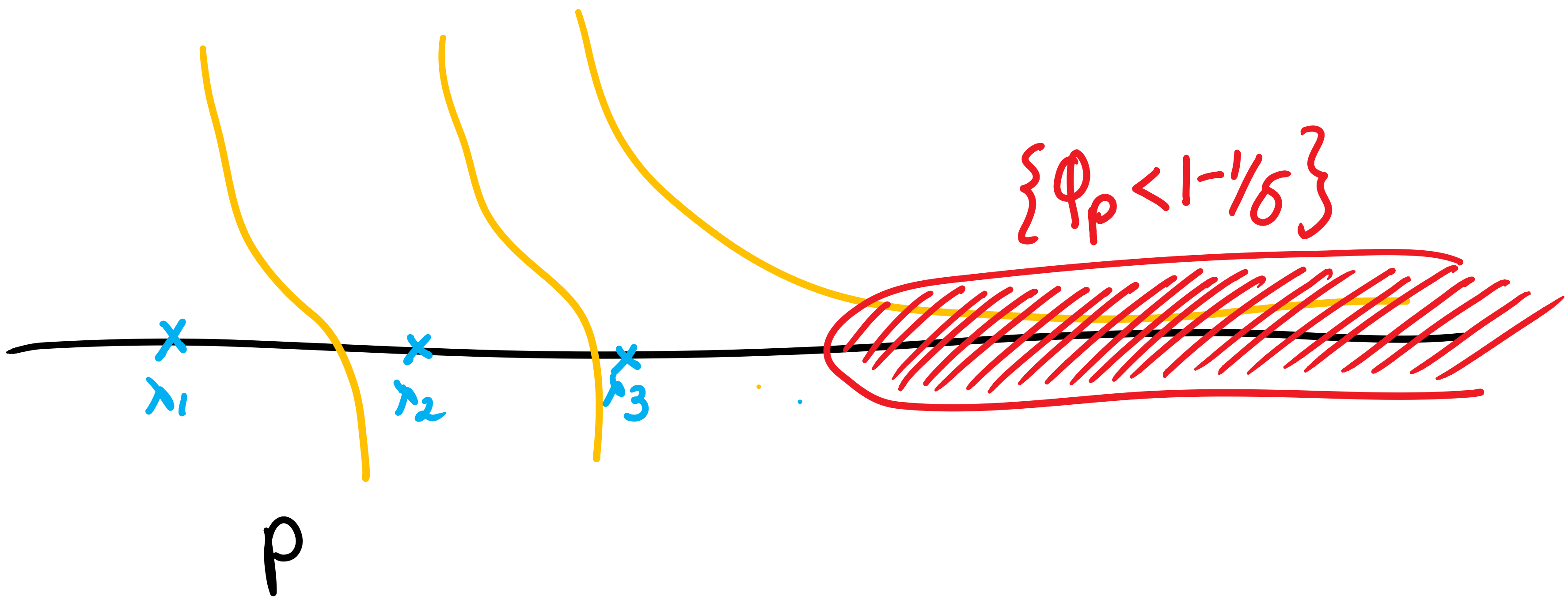

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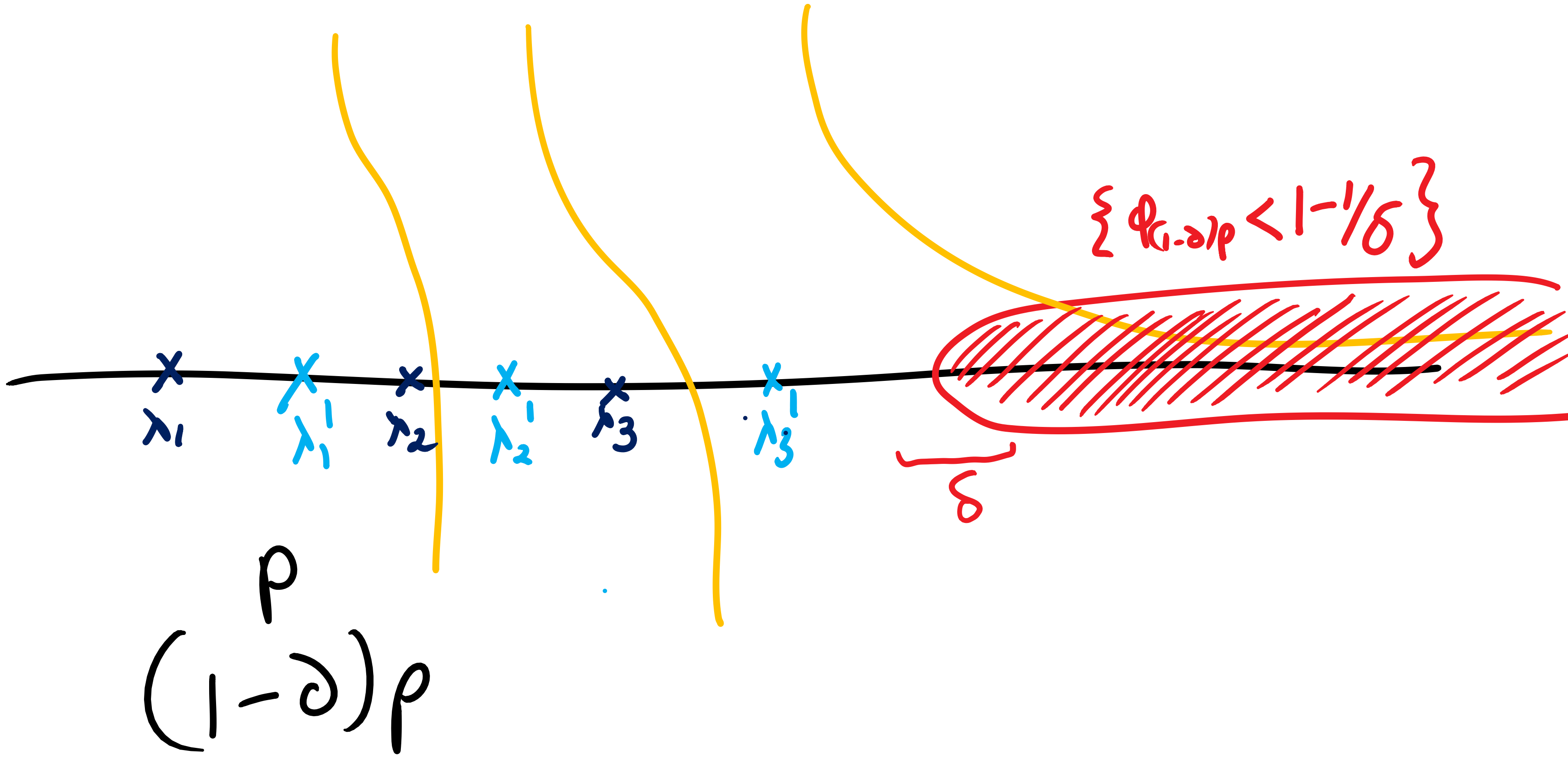

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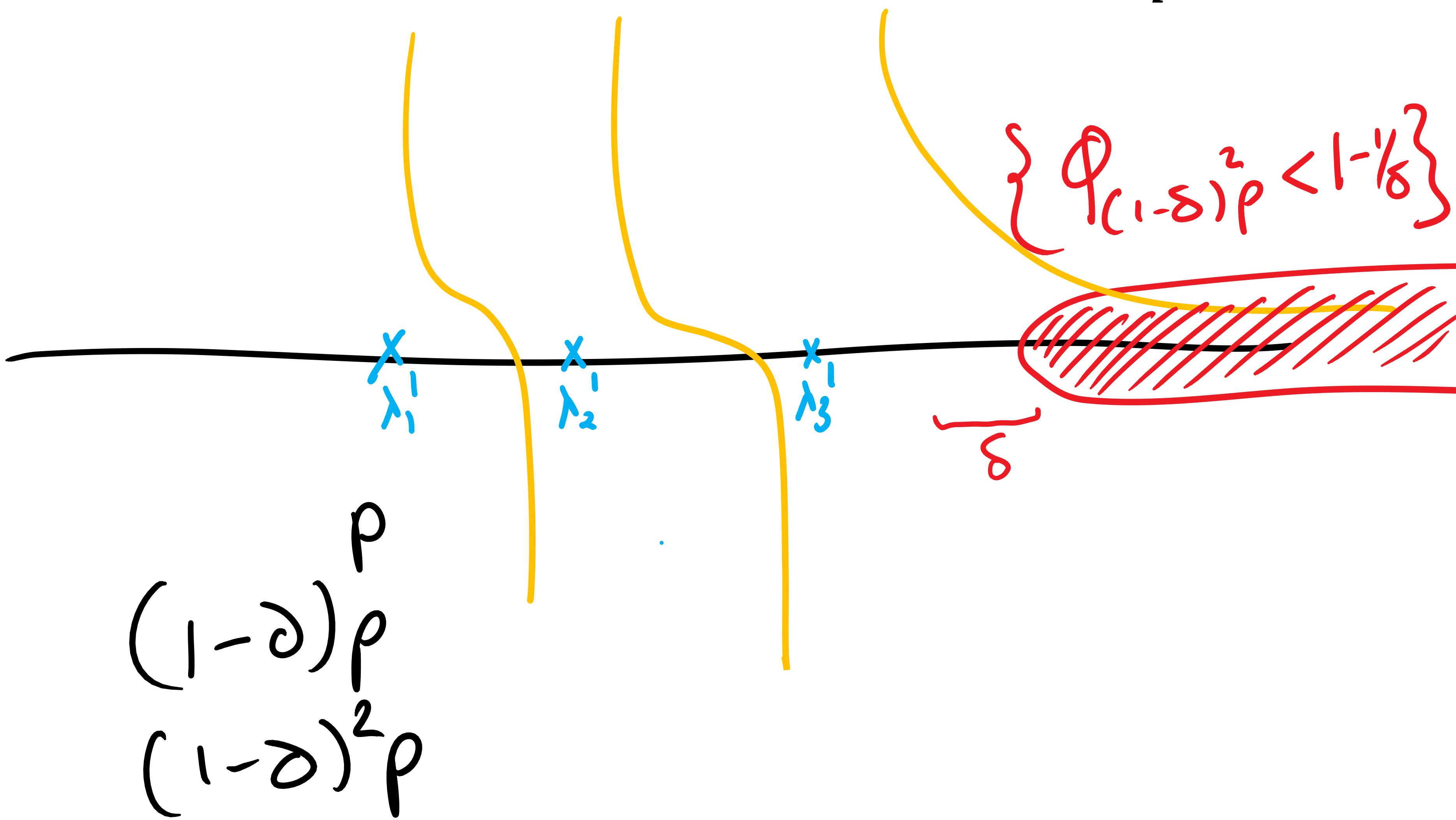

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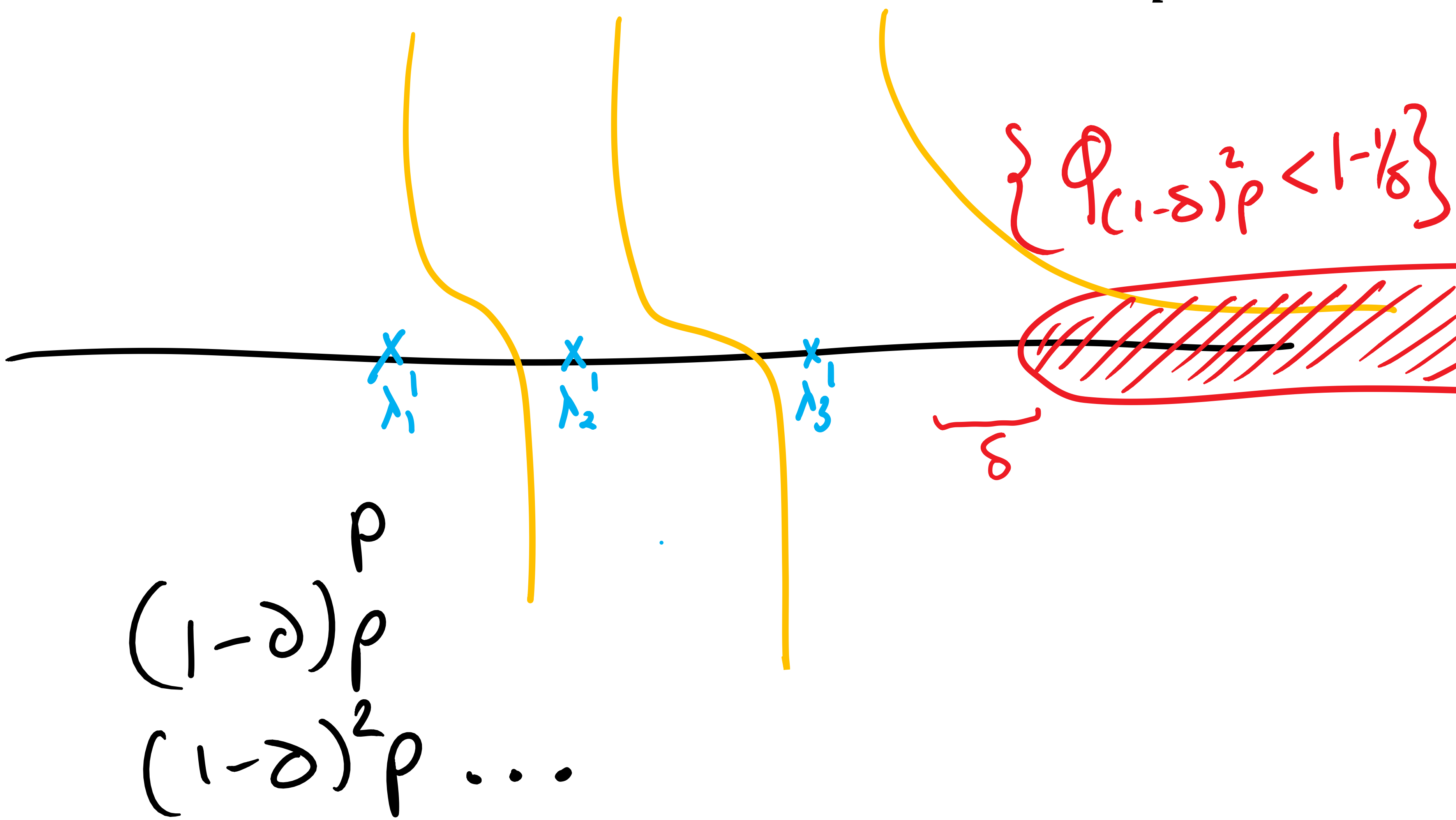

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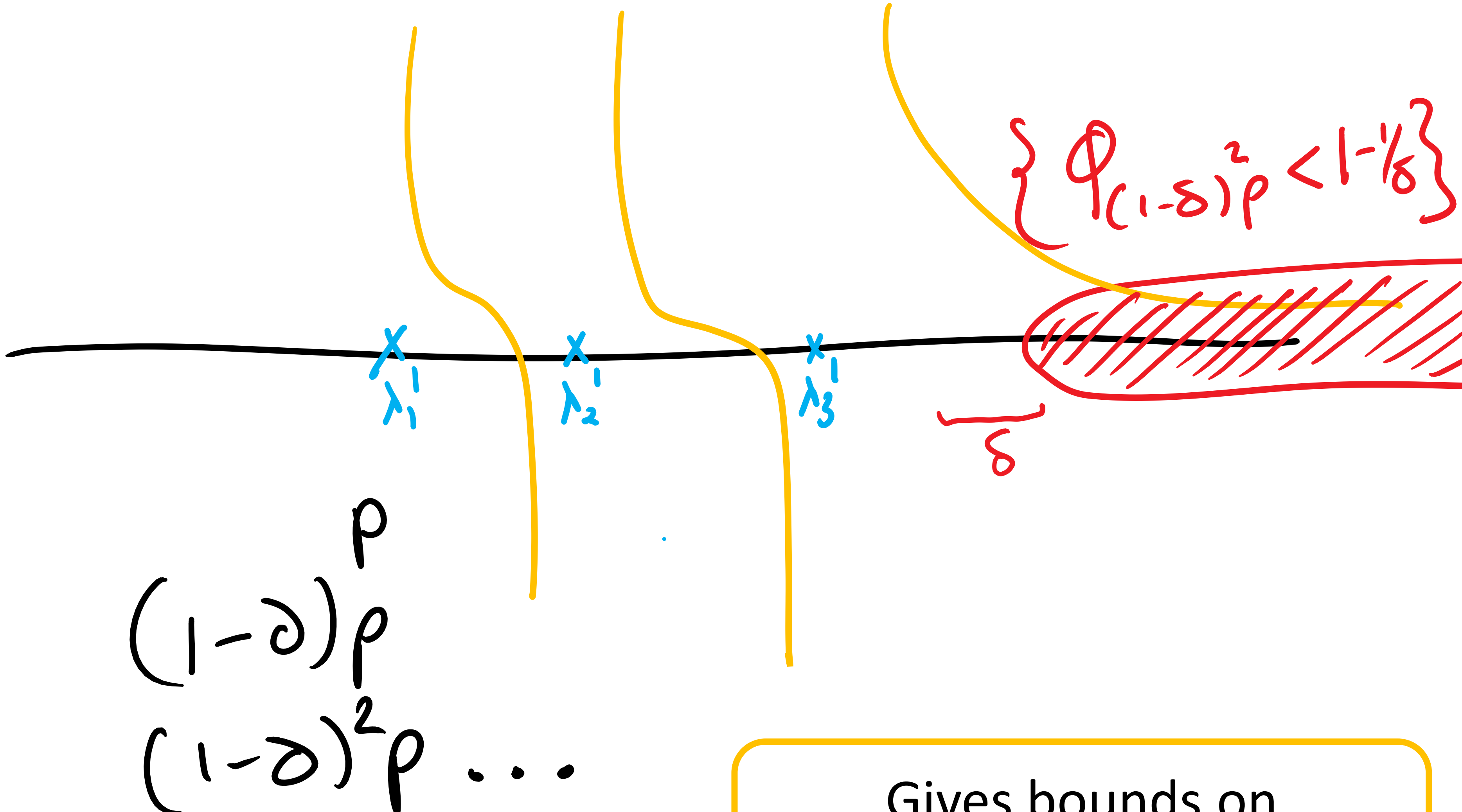



Gives bounds on
$\lambda_m(1-\partial)^k p(x)$

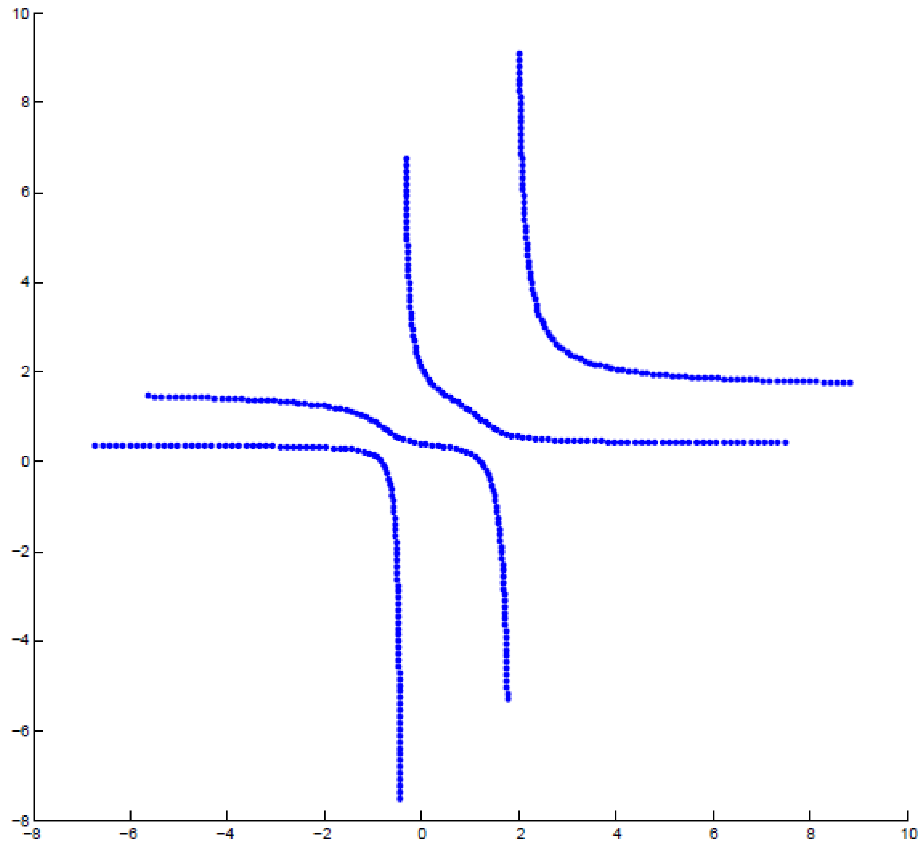
# The Bivariate Case

Given $p(x, y)$, I want to bound the roots of

$$(1 - \partial_x)(1 - \partial_y)p(x, y)$$

# The Bivariate Case

**Example**: roots of $p(x, y)$ real stable.



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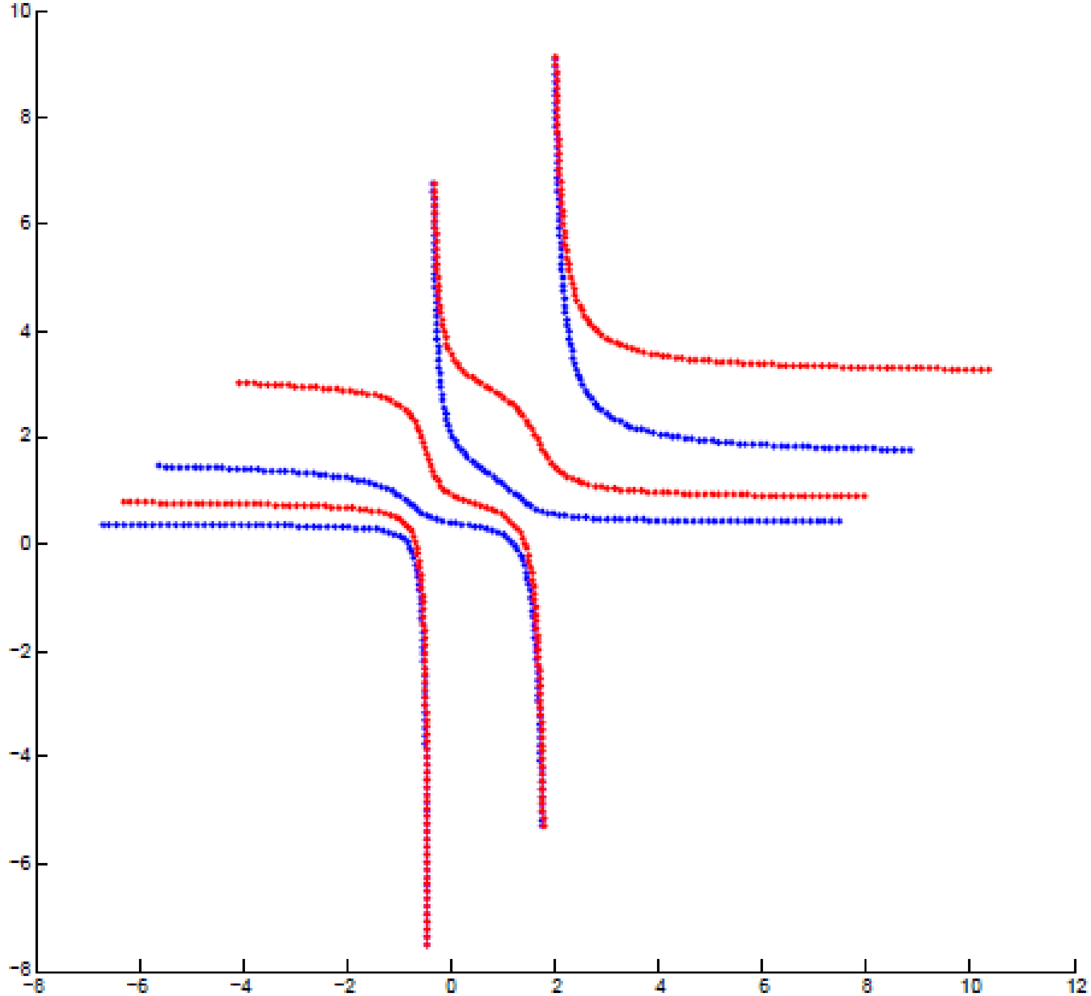

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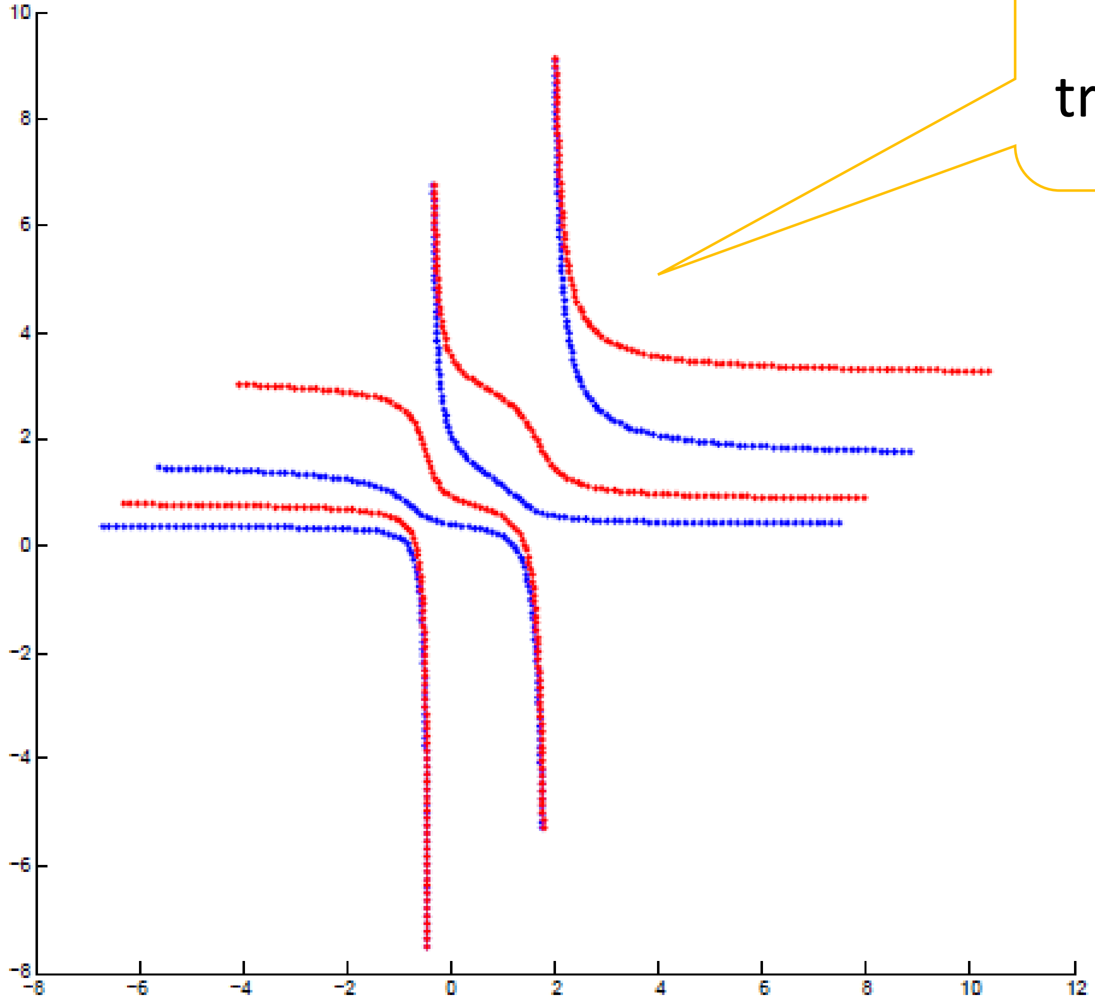



How to track these?

# The Bivariate Case

Given $p(x,y)$, I want to bound the roots of

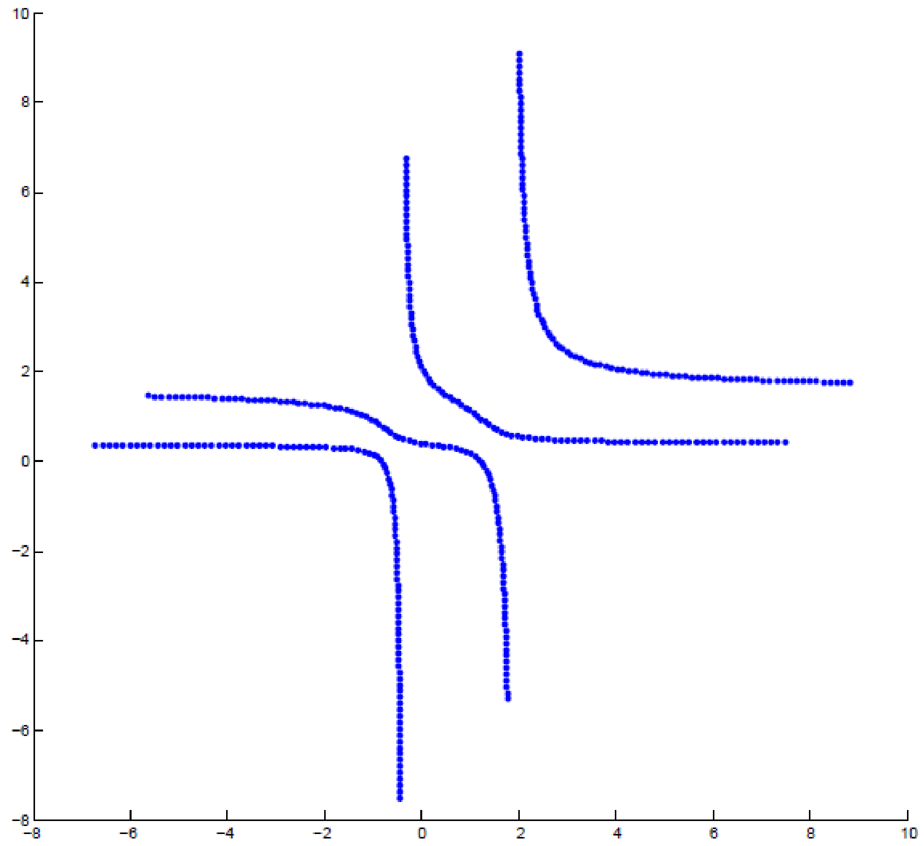
$$(1-\partial_x)\big(1-\partial_y\big)p(x,y)$$

Define **bivariate barrier function**

$$\Phi_p(x,y) = \left(\frac{\partial_x p(x,y)}{p(x,y)}, \frac{\partial_y p(x,y)}{p(x,y)}\right)$$

# Evolution of Bivariate $\Phi$ Level Sets

$p(x, y)$ real stable.



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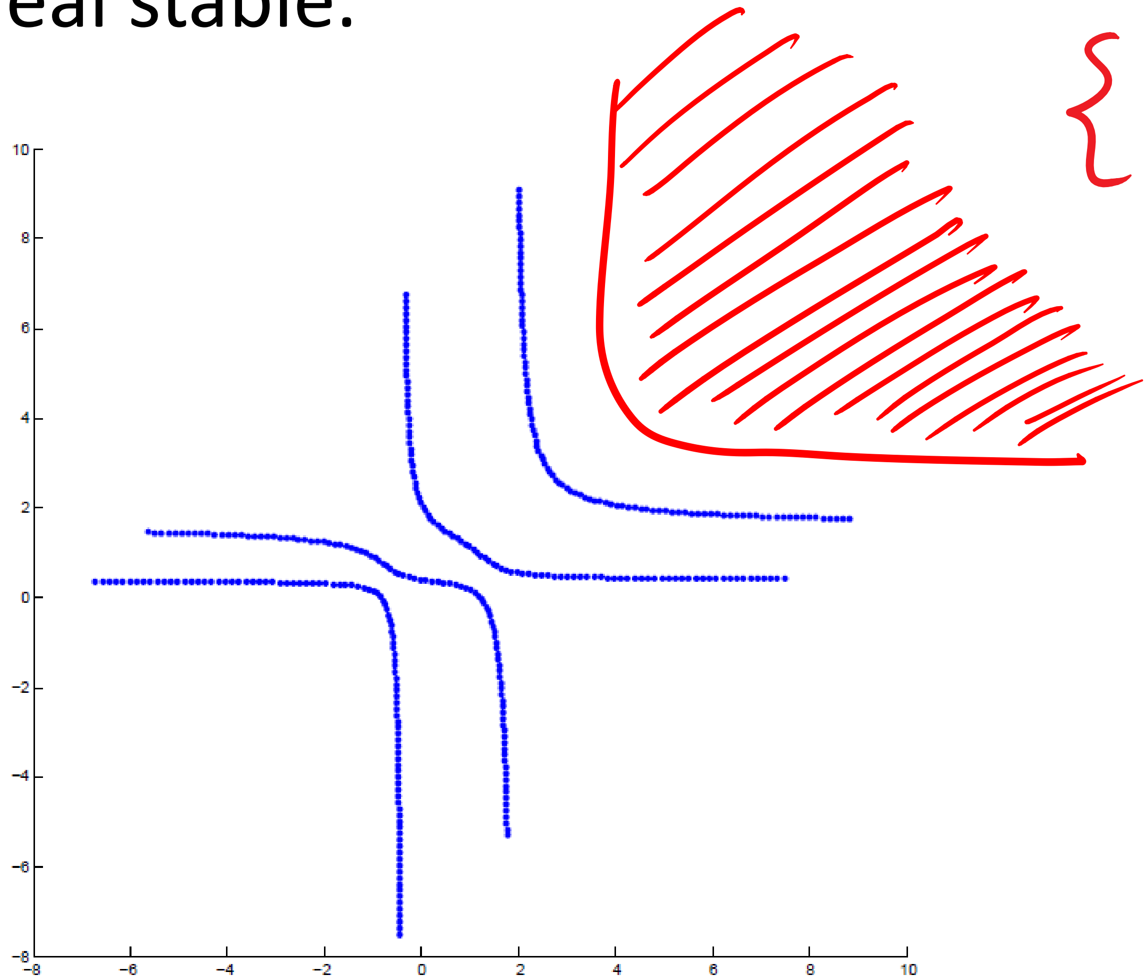
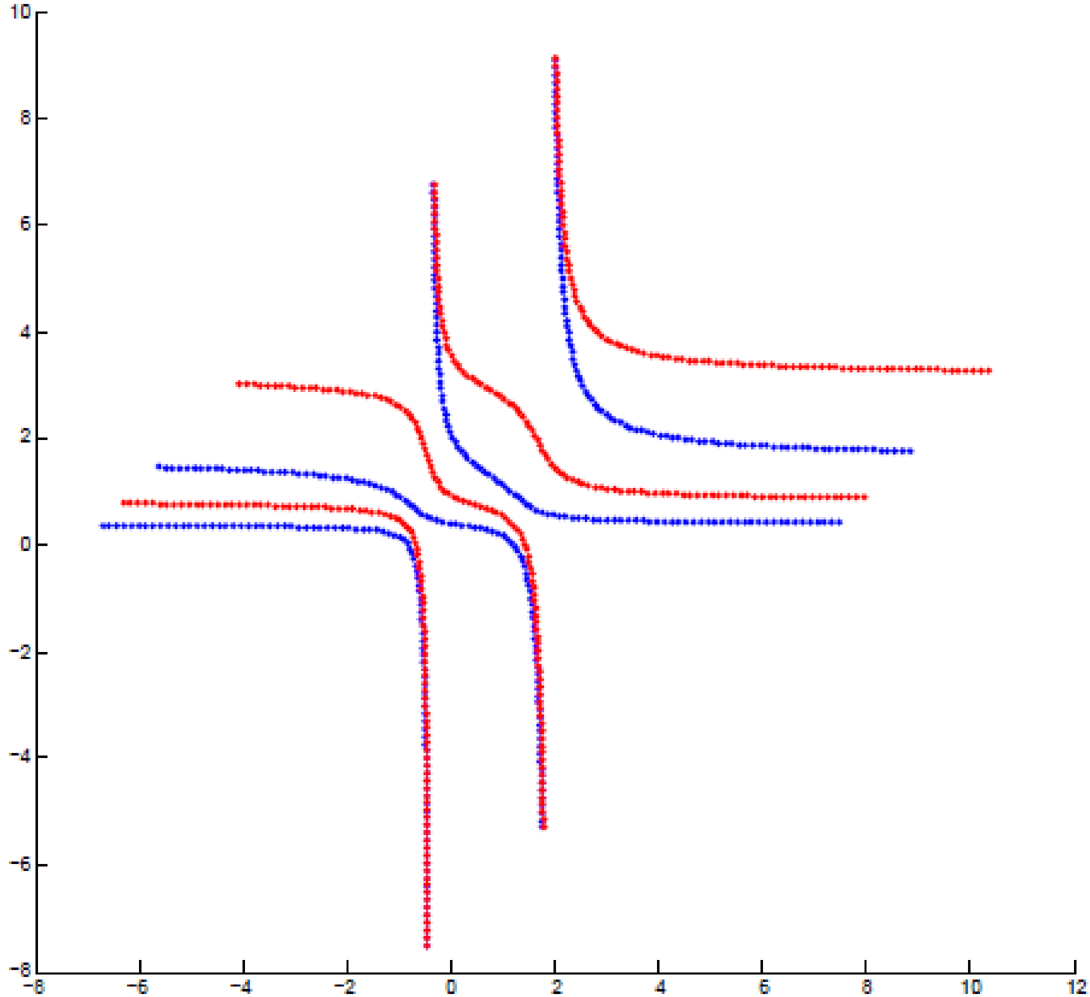



$\{\varphi_p < \cdot\cdot\}$

# The Bivariate Case

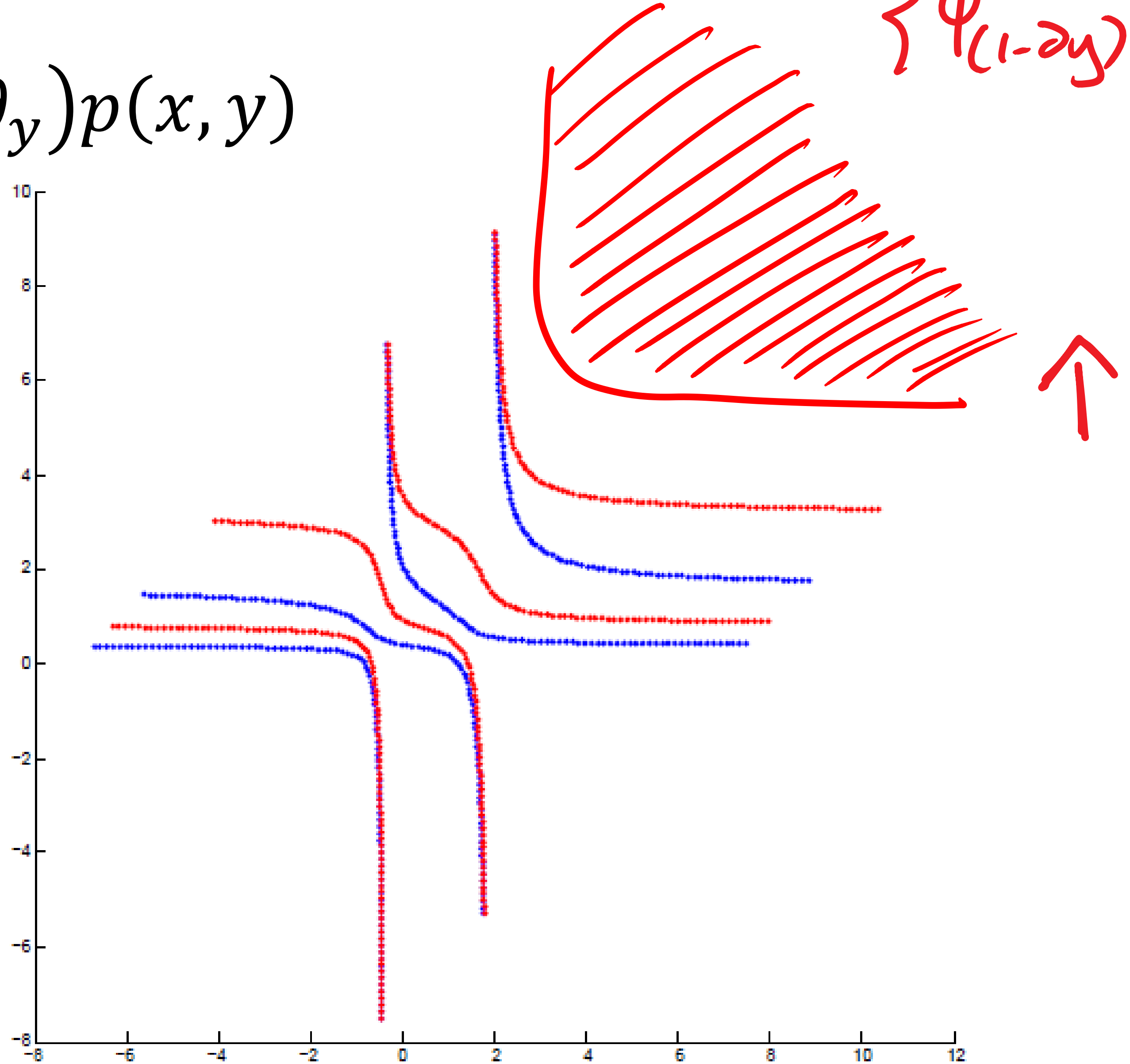
$$\left(1 - \partial_y\right)p(x, y)$$



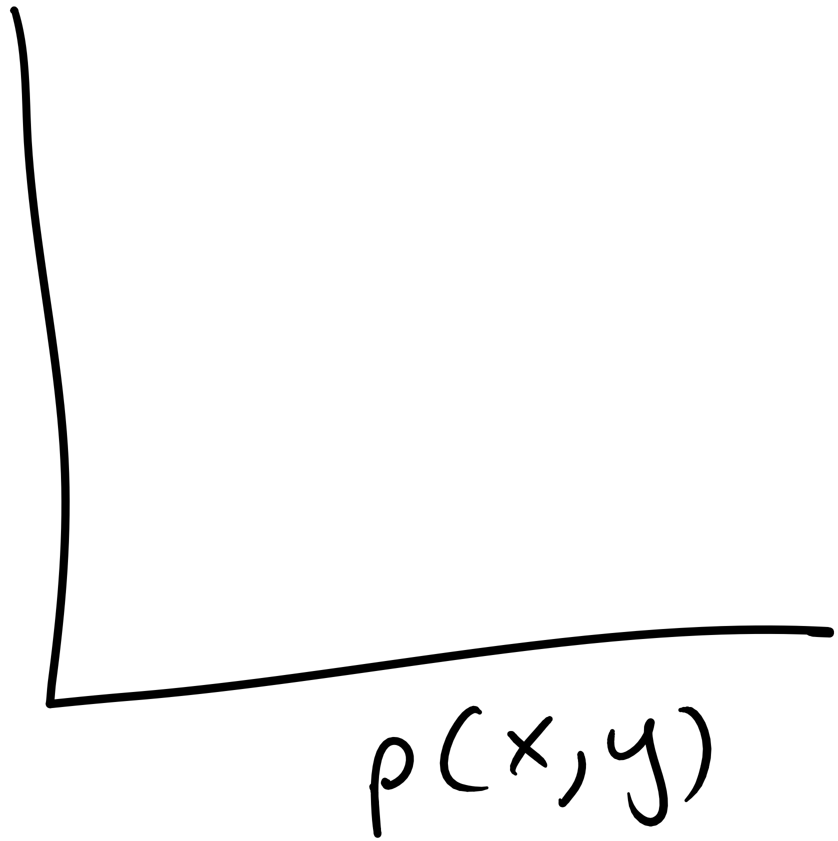
# The Bivariate Case

$\{\phi_{(1-\partial_y)p} < ..\}$

$$(1-\partial_y)p(x,y)$$



# Several Perturbations



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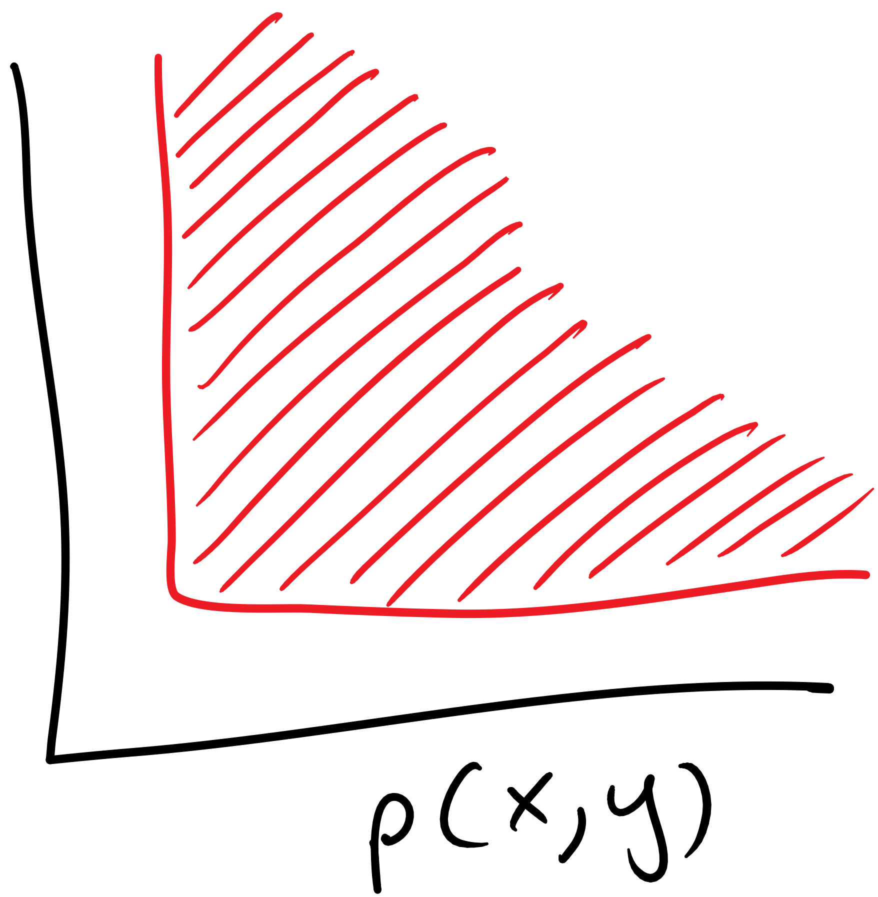

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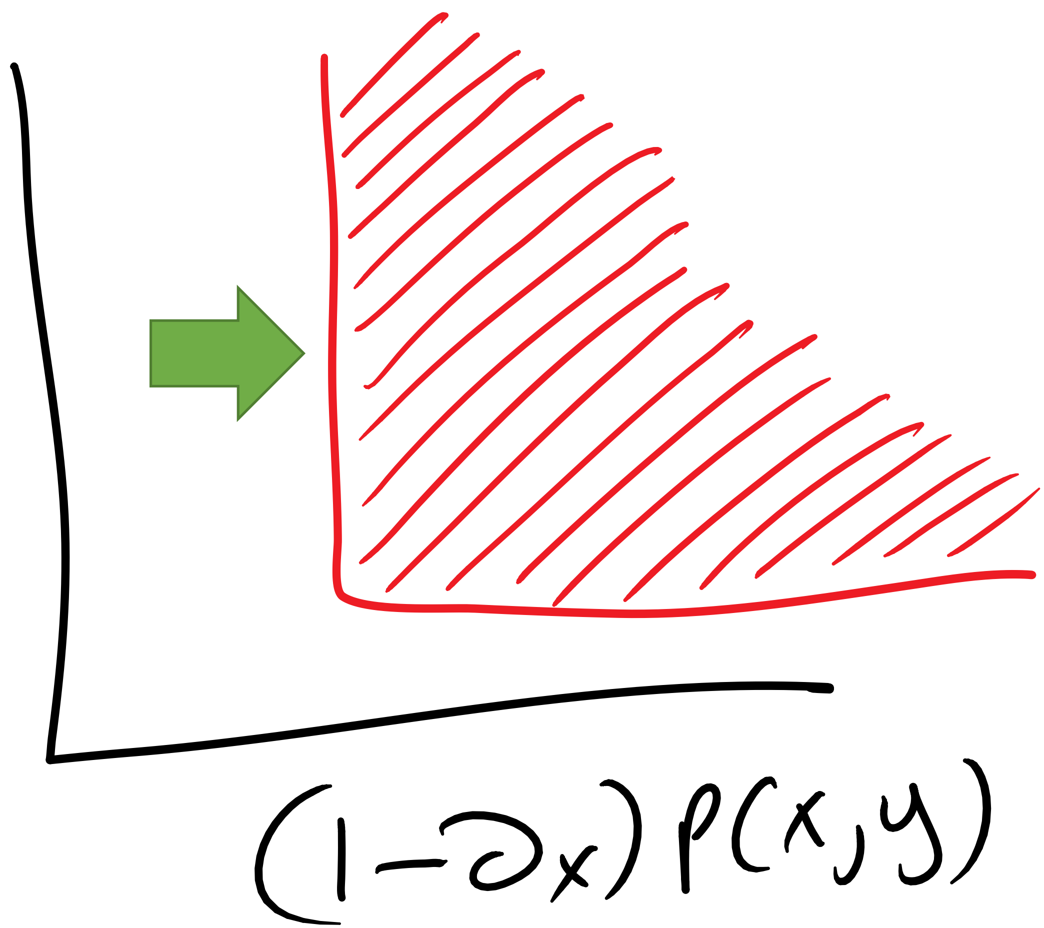

# Several Perturbations

$$(1-\partial_x)\,\rho(x,y)$$

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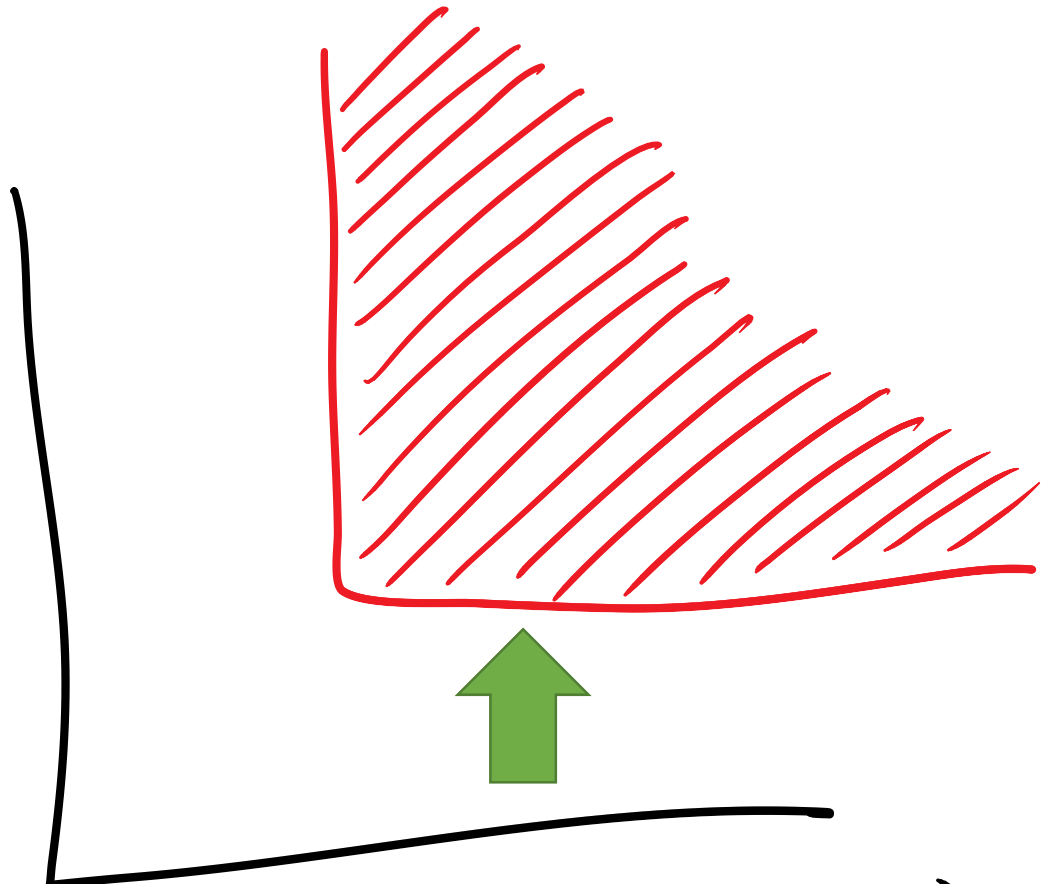

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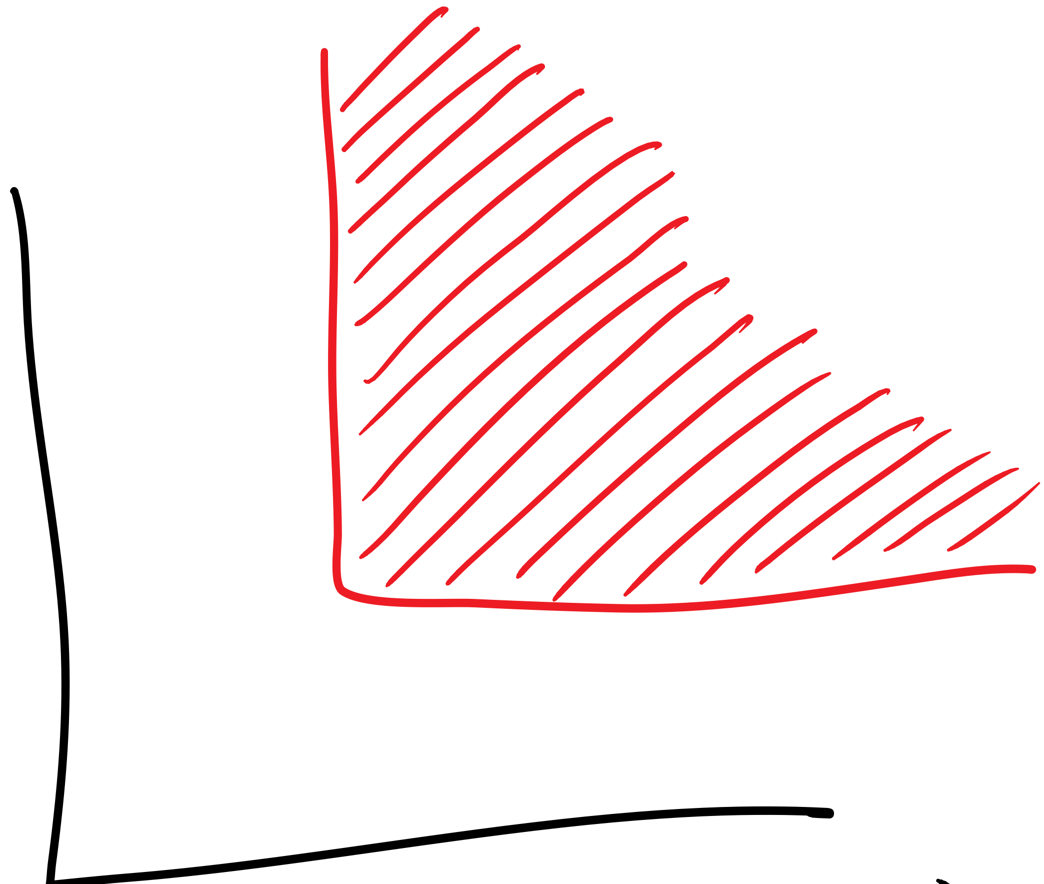

$(1-\partial y)(1-\partial x)\, p(x,y)$

....

# Key Ingredient

**Helton-Vinnikov'92:** All bivariate real stable polynomials are determinants:

$$p(x, y) = \det(xA + yB + C)$$

$$\boxed{\text{with } A \succcurlyeq 0, B \succcurlyeq 0}$$

This implies that the bivariate barrier has the same properties as the univariate one, and the old proof goes through.

# Basic Principle

Can track the evolution of multivariate zeros under $(1 - \partial_z)$ operators by studying related rational functions.

A quantitative version of the stability preserving property.

# End Result

If $Tr(A_i) \leq \epsilon$ and $\sum_i A_i = I$ then

$$\prod_{i=1}^{m} \left(1 - \partial_{z_i}\right) \det\left(xI + \sum_i z_i A_i\right)\Bigg|_{z_1 = \ldots z_m = 0}$$

Has roots bounded by $(1 + \sqrt{\epsilon})^2$

# 3-Step Plan

1. Show that there exist $v_1, \dots, v_m$ with

$$\lambda_{max}\, \chi\left(\sum_i v_i v_i^T\right) \leq \lambda_{max} \mathbb{E} \chi\left(\sum_i v_i v_i^T\right) \quad \checkmark$$

2. Calculate

$$\mathbb{E}\chi =: \mu(x) = \prod_{i=1}^{m} \left(1 - \partial_{z_i}\right) \det\left(xI + \sum_i z_i A_i\right)\Bigg|_{z_1 = \cdots 0} \quad \checkmark$$

3. Bound the largest root $\lambda_{max} \mu(x) \leq 1 + \sqrt{\epsilon}$ $\checkmark$

Assuming $\mathbb{E} \sum_i v_i v_i^T = I$ and $||v_i||^2 \leq \epsilon$

# Main Theorem

If $v_1, \ldots, v_m \in \mathbb{R}^n$ are **independent**,

$$\mathbb{E} \sum_i v_i v_i^T = I \text{ and } ||v_i||^2 \leq \epsilon$$

Then

$$\mathbb{P}\left[\left\|\sum_i v_i v_i^T\right\| \leq 1 + O(\sqrt{\epsilon})\right] > 0$$

# Spectral Discrepancy Theorem

Suppose $v_1, \ldots, v_m \in \boldsymbol{R}^n$ are vectors $||v_i||^2 \leq \epsilon$ and

$$\boxed{\sum_i v_i v_i^T = I_n}$$

Then there is a partition $T_1 \cup T_2$ such that

$$\boxed{\left(\frac{1}{2} - 5\sqrt{\epsilon}\right) I \preccurlyeq \sum_{i \in T_j} v_i v_i^T \preccurlyeq \left(\frac{1}{2} + 5\sqrt{\epsilon}\right) I}$$

# Open Questions

Quantitative analysis of other stability-preserving operators.

More applications of discrepancy theorem.

Algorithms.

# Two Tools

**Matrix-Determinant Lemma:**

$$\begin{aligned}\det(M + vv^T) &= \det(M)\det(I + M^{-1}vv^T) \\ &= \det(M)(1 + v^T M^{-1} v)\end{aligned}$$

# Two Tools

**Matrix-Determinant Lemma:**

$$\det(M + vv^T) = \det(M)\det(I + M^{-1}vv^T)$$
$$= \det(M)(1 + v^T M^{-1} v)$$

**Jacobi's Formula:**

$$\partial_t \det(M + tA) = \det(M)\, \partial_t(\mathrm{I} + \mathrm{M}^{-1}\mathrm{A})$$
$$= \det(M)\, Tr(M^{-1}A)$$

# $(1-\partial_z)$ operators = rank-1 updates

$$\mathbb{E}\det(M - vv^T) = \mathbb{E}\det(\mathrm{M})(1 - v^T M^{-1} v)$$

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$$= (1 - \partial_t)\det(M + t\mathbb{E}vv^T)\,|_{t=0}$$

# Proof of Central Identity

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$\square$