CMSC 39600 - Lec 14 (Nov 13)

Today

Plane-Point Test

Admin

- Extra lecture
- PSet 1
- PSet 2.
- Guest Lecture

Last Lecture

$$\mathcal{A} : S_k^{k+1} \longrightarrow P_k^d$$

(affine spaces of dim k in $\mathbb{F}^{k+1}$) $\quad$ k-variate poly of deg $\leq d$.

$G_A = (V_A, E_A)$

$V_A = S_k^{k+1}$

$E_A = \{ (s_1, s_2) \mid \forall x \in s_1 \cap s_2, \; \mathcal{A}(s_1)(x) = \mathcal{A}(s_2)(x) \}$

(If $s_1 \cap s_2 \neq \phi$ & $(s_1, s_2) \in E$

then $\mathcal{A}(s_1)|_{s_1 \cap s_2} \equiv \mathcal{A}(s_2)|_{s_1 \cap s_2}$

Lemma: For $k \geq 3$.

$(s_1, s_2) \notin E_A$

$$\Pr_{s \in V} \left[(s, s_1) \in E_A \wedge (s, s_2) \in E_A \right] \leq \frac{d+1}{q} = \frac{2d}{q}$$

Lemma 2 $\quad \epsilon = \frac{2d}{q}$

$\exists$ partition of vertices $V = \coprod_{i=1}^{t} V_i$ st

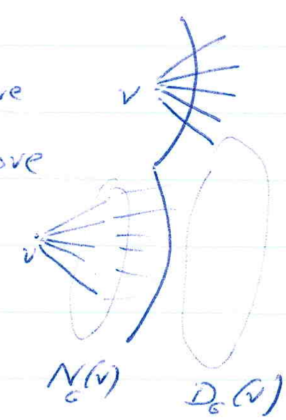
(a) $\forall\ i, \quad |V_i| = 1$ or $|V_i| > 2\sqrt{\epsilon}\,|V|$

(b)

$$\Pr_{b_1, b_2}\left[(b_1, b_2) \notin E_A \ \vee\ \exists i \ \ b_1, b_2 \in V_i \right] \geq 1 - 5\sqrt{\epsilon}$$

Pf: Pick a vertex v

- if $d_G(v) \leq 2\sqrt{\epsilon}\,|V|$, remove
- if $d(v) > 2\sqrt{\epsilon}\,|V|$, remove



At end left with isolated vertices or cliques of size $\geq 2\sqrt{\epsilon}\,|V|$

$v_1 \ldots v_\ell$ - list of vertices picked

$G_1 \ldots G_\ell$

$I_1 = \{ i \mid d_{G_i}(v_i) \leq 2\sqrt{\epsilon}\,|V(G_i)| \}$

$|I_1| < |V|$

$\sum |N_{G_i}(v_i)| \leq 2\sqrt{\epsilon}\,|V_i|\,|I_1| \leq 2\sqrt{\epsilon}\,|V|^2.$

$I_2 = [\ell] \setminus I_1$

$\sum |N_{G_i}(v_i)| \cdot |D_{G_i}(v_i)| \leq |V|$

$\sum |D_{G_i}(v_i)| \cdot \epsilon |V| < \sum |D_{G_i}(v_i)| \cdot \frac{|N_{G_i}(v_i)|}{2\sqrt{\epsilon}}\,\epsilon \leq \frac{\sqrt{\epsilon}}{2} |V|^2$

edges removed $\frac{5}{2}\sqrt{\epsilon}\,|V|^2$

pairs $5\sqrt{\epsilon}\,|V|^2$

Claim: (Large Cliques correspond to Poly)
For every clique $W \subseteq V$, $|W| \geq \left(\frac{2d+1}{9}\right)|V|$,
there exists a poly $Q: \mathbb{F}^k \to \mathbb{F}$ of deg $2d$
s.t. $\forall s \in W$

$$Q|_s \equiv A(s)$$

Exercise

Lemma 3: For any $\delta \geq \frac{20}{3}\sqrt{\frac{2d}{9}}$, there exists list of poly $Q_1 \ldots Q_t$ $(t \leq 4/\delta)$ of deg $\leq 2d$ s.t.

$$\Pr_{s_1, s_2 \in S_k^{k+1}} \left[(s_1, s_2) \notin E_A \vee \exists i \; Q|_{s_1} \equiv A(s_1) \wedge Q|_{s_2} \equiv A(s_2) \right] > 1-\delta$$

Pf: $S_1 \ldots S_\ell$ - small cliques $\left(|S_i| < \frac{\delta}{4}|V|\right)$

$$\sum |S_i|^2 < \frac{\delta}{4}|V| \sum |S_i| \leq \frac{\delta}{4}|V|^2.$$

$L_1 \ldots L_t$ - large cliques $(t \leq 4/\delta)$
$Q_1 \ldots Q_t$ - corresponding poly

$$\Pr_{s_1 s_2}\left[(s_1, s_2) \notin E_A \vee \exists i: s_1 s_2 \in L_i\right] > 1 - 5\sqrt{\frac{2d}{9}} - \frac{\delta}{4}$$

$$> 1 - \frac{3}{4}\delta - \frac{\delta}{4} = 1-\delta$$

Hence, proved.

Plane Space-point to Space-Space

$\Pr [A(b)(x) = A(x)] \geq \gamma \implies \Pr[$

$$\mathop{\mathbb{E}}_{a \in S^{k+1}_{k-1}} \left[\mathop{\mathbb{E}}_{\substack{(b_1, b_2) \in S^{k+1}_{k} \\ b_1 \cap b_2 = a}} \left[\Pr_x [A(b_1)(x) = A(x) = A(b_2)(x)] \right] \right] \geq \gamma^2 - \frac{1}{q}.$$

List Decoding to Decoding

Suppose $\Pr_{A, b, x} [A(b)(x) = A(x)] \geq \gamma \geq$

$$\Pr [A(b)(x) \neq A(x) \vee \exists i \;\; f_i|_b = A(b)] \geq 1 - \delta$$

$$\Downarrow$$

For all $\varepsilon \geq \sqrt[3]{\frac{t}{q}}$, there exists a poly f_i

$$\Pr [A(x) = f_i(x)] \geq \gamma - \delta - 2\varepsilon$$

Pf: Suppose otherwise that for all i

$$\mathrm{agr}(f_i, A) < \gamma - \delta - 2\varepsilon.$$

A_i = regions of agreement

$$\mu_i = \frac{|A_i|}{|\mathbb{F}^m|} < \gamma - \delta - 2\varepsilon.$$

$$\Pr \left[\frac{|b \cap A|}{|b|} \geq \mu_i + \varepsilon \right] \leq \frac{\mu_i}{q \varepsilon^2} \leq \frac{\varepsilon}{t}$$

$$\mathrm{Var}_b \left[\frac{|b \cap A|}{|b|} \right] \leq \frac{\mu}{q} \leq \frac{1}{q}$$

Union Bd

$$\Pr \left[\exists i \;\; \frac{|b \cap A_i|}{|b|} \geq \gamma - \delta - \varepsilon \right] \leq \varepsilon.$$

S bad if $\exists i \ |S \cap A_i| \geq (\gamma - \delta - \varepsilon)|S|$

$\Pr(\text{bad space}) \leq \varepsilon$.

$$\Pr[\mathcal{A}(S)(x) = x \wedge \exists i \ \mathcal{A}(S) \equiv f_i|_S]$$

$$\leq \Pr[(\text{bad space})] + \Pr[\underset{= \mathcal{A}(x)}{\mathcal{A}(S)(x)} \mid S\text{-good}]$$

$$\leq \varepsilon + \Pr[\mathcal{A}(S)(x) = \mathcal{A}(x) \mid S\text{ good} \wedge \exists i]$$

$$\leq \varepsilon + (\gamma - \delta - \varepsilon)$$

$$= \gamma - \delta$$

However $\Pr[\mathcal{A}(S)(x) = \mathcal{A}(S)] \leq (\gamma - \delta) + \delta \leq \gamma$ (contradiction)

Lemma:

If $\tilde{\mathcal{A}}$:
- Pick a
- Pick S_1 S_2 s.t. $S_1 \cap S_2 = a$.
- If $\exists i$ $Q_i|_{S_1} \equiv \mathcal{A}(S_1)$ & $Q_i|_{S_2} \equiv \mathcal{A}(S_2)$
  - then output $Q_i|_a$.
- else output $\perp$.

$$\Pr_{a,x}[\tilde{\mathcal{A}}(a)(x) \neq \mathcal{A}(x) \vee \exists i \ Q_i|_a \equiv \tilde{\mathcal{A}}(a)] = 1.$$

$$\Pr[\tilde{\mathcal{A}}(a)(x) = \mathcal{A}(x)] \geq \gamma^2 - 2\delta$$

then done.

$\exists i$, $\Pr[\mathcal{A}(x) = Q_i(x)] \geq \gamma^2 - 2\delta - \delta = \underline{\underline{\gamma^2 - 3\delta}}$.

$$\Pr[\tilde{A}(a)(x) = A(x)] \geq r^2 - 2\delta.$$

$a, \quad b_1, b_2 \qquad x.$

$$\Pr[(b_1, b_2) \notin E_A \ \vee \ \exists i \ \ Q_i|_{b_1} = A(b_1) \wedge Q_i|_{b_2} = A(b_2)] \geq 1 - \delta - \frac{1}{q}.$$

We want $\Pr[A(b_1)(x) = A(x) = A(b_2)(x) \ \wedge \ \exists i \qquad] \geq r^2 - 2\delta$

$$\Pr[\qquad] \geq r^2 - \frac{1}{q}.$$

$$\Pr[\qquad \wedge \ \neg\exists i \qquad]$$

$$= \Pr[(b_1, b_2) \in E_A \wedge \neg\exists i \qquad] + \Pr[(b_1, b_2) \notin E_A \wedge \qquad]$$

$$\leq \delta + \frac{1}{q} + \frac{d}{q}$$