1. [Corruption bound and inner product] (8+7)

(a) Let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ be a Boolean function and $\mu$ be a distribution on $\mathcal{X} \times \mathcal{Y}$ such that for every rectangle $R = S \times T \subseteq \mathcal{X} \times \mathcal{Y}$ with $\mu(R) > \rho$, we have that

$$\mu(R \cap f^{-1}(1)) > \varepsilon \cdot \mu(R \cap f^{-1}(0)).$$

In other words, every large rectangle is 1-corrupted. Then, prove that for every $\delta > 0$, we have

$$2^{R_{\rho}(f)} \geq \frac{1}{\rho} \left( \mu(f^{-1}(0)) - \frac{\delta}{\varepsilon} \right).$$

(b) Using the above corruption bound or otherwise, prove that

$$R_{\frac{1}{2} - \varepsilon}(\text{IP}) \geq n - O\left( \log \frac{1}{\varepsilon} \right) - O(1).$$

2. [Private coins vs. Public coins for the zero-error case] (18)

Prove that for any $f : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$, we have

$$R_0(f) = O\left( R_{0}^{\text{pub}}(f) + \log n \right).$$

3. [Augmented Index function] (10)

Consider the following 2-party randomized one-way communication problem $\text{alND}$. Alice is given a $n$ bit string $x \in \{0, 1\}^n$ while Bob is given an index $i \in [n]$ and the first $(i-1)$ bits of $x$, i.e., $x_1, \ldots, x_{i-1}$. The goal of the protocol is for Bob to determine $x_i$. Prove that the lower bound proved in class for the index function continues to hold even for this augmented version of the problem. More precisely, show that

$$R_{\frac{1}{2} - \varepsilon}^A \rightarrow_B (\text{alND}) \geq 2 \log \varepsilon \cdot \varepsilon^2 n.$$
4. [Multi-party disjointness] (5+10+5)
Recall the proof of the multi-party disjointness UDISJ proved in lecture. We showed that the private coins randomized communication complexity of UDISJ_{n,t} in the number-in-hand broadcast model is $\Omega(n/t^2)$. An intermediate step in the proof of this result involved showing the following geometric inequality:
\[
\sum h^2(\Pi_{\mathcal{T}}, \Pi_{v_i}) \geq t \cdot h^2(\Pi_{\mathcal{T}}, \Pi_{\mathcal{T}}),
\]
where $h^2(\cdot, \cdot)$ denotes the squared-Hellinger distance and $\Pi_{z}$ denotes the transcript distribution on input $z$. In this problem, we will improve the lower bound to $\Omega(n/t)$ by improving the above geometric inequality to
\[
\sum h^2(\Pi_{\mathcal{T}}, \Pi_{v_i}) \geq O(1) \cdot h^2(\Pi_{\mathcal{T}}, \Pi_{\mathcal{T}}),
\]
(a) Prove that for any $n + 1$ vectors $v_0, v_1, \ldots, v_n$, we have
\[
\sum_{i=1}^{n} \|v_i - v_i\|^2 \geq \frac{1}{n} \sum_{1 \leq i < j \leq n} \|v_i - v_j\|^2.
\]
(b) Suppose $A_1, \ldots, A_n$ are pairwise disjoint collection of $n = 2^k$ subsets of $[t]$. Let $A = \bigcup_i A_i$. Show that
\[
\sum_{i=1}^{n} h^2(\Pi_{\emptyset}, \Pi_{A_i}) \geq h^2(\Pi_{\emptyset}, \Pi_{A}) \prod_{l=1}^{k} \left(1 - \frac{1}{2^l}\right).
\]
(c) Conclude that the private coins randomized communication complexity of UDISJ_{n,t} in the number-in-hand broadcast model is $\Omega(n/t)$.

5. [streaming algorithms for Dyck variations?] (12)
Consider the following variation of the context-free grammar that generates 2-Dyck language:
\[
S \rightarrow SS, (S), (S), [S], [S], \varepsilon.
\]
Let $L$ be the language generated by the above grammar. Note that $L$ is the set of well-formed parentheses of 2-types with the modification being that ( can be closed by either ) or ] while [ can be closed only by ]. Show that any $r$-pass (randomized) streaming algorithm for $L$ requires space at least $\Omega(n/r)$.

6. [Aborting index function problem] (10)
Suppose $\Pi$ is a randomized one-round protocol (Alice send a message, Bob guesses) for the index function problem with satisfying the following conditions. Let $p \in [0, 1]$ and $q \in [\frac{1}{2}, 1]$. For every input,
(a) Bob may abort without giving a 0-1 answer, but he may do so with probability at most $1 - p$;
(b) Conditioned on Bob declaring the answer, the probability that the answer is correct is at least $q$.
Show that Alice must send at least $p(1 - H(q))n$ bits in any such protocol.

7. [Chasing the pointer’s head] (15)
Consider the following variant of the pointer chasing problem $P_k$. There are $k + 1$ layers of vertices: $L_0, L_1, \ldots, L_k$, ($L_0$ has only one vertex $v_0$). Pointers go from one layer to the next and define a unique path $v_0, v_1, \ldots, v_k$ from $L_0$ to $L_k$. Alice has the pointers leaving the even layers and Bob has the pointers leaving the odd layers. The goal is to determine the msb of $v_k$. Suppose Alice starts the communication. Show that there is an $O(kn)$-bit $\left\lceil \frac{k}{2} \right\rceil$-round deterministic protocol for this problem.