Problem Set 1

- Due Date: **8 Feb (Tue), 2018**
- Turn in your problem sets electronically (LaTeX, pdf or text file) by email. If you submit handwritten solutions, start each problem on a fresh page.
- Collaboration is encouraged, but all writeups must be done individually and must include names of all collaborators.
- Refering sources other than the text book and class notes is strongly discouraged. But if you do use an external source (eg., other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.
- The points for each problem are indicated on the side.
- Be clear in your writing.

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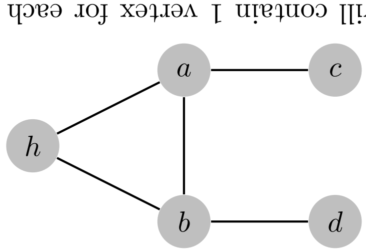
1. **[P, NP, coNP, NP-complete, coNP-complete]** (10+3+6+6)

   Identify the smallest complexity class (among P, NP, co-NP) in which each of the problem lies. Furthermore, if the problem is in NP (or co-NP), mention if the problem is NP-complete or coNP-complete. In each case, substantiate your classification.

$$\begin{aligned}
\text{3-C{\small OLOUR}} &= \{G \mid G \text{ can be coloured using 3 colours such that no edge is monochromatic }\}\\
\text{NON-ISO} &= \{(G,H) \mid \text{ there is no edge preserving bijection between } G \text{ and } H\}\\
\text{2SAT} &= \{\varphi \mid \varphi \text{ has at most 2 literals in any clause and is satisfiable. }\}\\
\text{BIPARTITE} &= \{G \mid G \text{ is a bipartite graph}\}
\end{aligned}$$

   [Hint: for 3-Colour: Reduce from 3-SAT. Your graph will contain 1 vertex for each literal, and 3 special vertices connected in a triangle (which must then be coloured with the three distinct colours, say T, F and Green). You may find the following gadget (see graph below) useful. Suppose that the head h and the two legs c and d are connected to the special vertex coloured Green, then any valid 3-colouring of this graph has the property that at least one of the two legs c or d has the same colour as the head h.]



2. [**VERTEX COVER**] Problems 2.15 and 2.34 (10+(4+2))

   [Hint: For 2.34, you may use the fact that there exists a polynomial time algorithm for matching.]

3. [**coNP**] Problems 2.24, 2.25 and 2.33 (5+5+8)

4. [**Unary languages and NP**] Problems 2.30 and 2.32 (8+7)

   [Hint: Problem 2.30: Besides the hint given in the textbook, use the fact that for any m, the number of strings in a unary language L of length at most m is at most m.]

5. [**PRIMES $\in$ NP**] Problem 2.5 (10)

6. [Circuit-SAT**is NP-complete**] (8+8)

   A circuit C on n inputs is a directed acyclic graph with n *sources* (vertices with no incoming edges) and one *sink* (vertex with no outgoing edges). All nonsource vertices are called *gates* and are labeled with one of $\vee, \wedge$ or $\neg$. The vertices labeled with $\vee$ or $\wedge$ have fan-in 2 and the vertices labeled with $\neg$ have fan-in 1. If C is a Boolean circuit and $x \in \{0,1\}^n$ is some input, then the output of C on x, denoted by $C(x)$, is defined in the natural way. A circuit C is said to be satisfiable if there exists a x such that $C(x) = 1$.

   $$\text{Circuit-SAT} = \{C \mid C \text{ is satisfiable}\}$$

   (a) Prove (along the lines of the proof of the Cook-Levin Theorem discussed in lecture) that Circuit-SATis NP-hard. (don't reduce SAT to Circuit-SAT).

   [Hint: Consider a binary encoding of every snapshot z_i. Show that there exists a circuit that computes snapshot z_i given snapshots $z_i, z_{\text{prev}(i)}$ and y. Compose these circuits suitable to get the final result.]

   (b) Prove that Circuit-SAT$\leq_p$ 3SAT (this gives an alternate proof of the Cook-Levin Theorem).