Problem Set 2

- Due Date: **20 Feb (Tue), 2018**
- Turn in your problem sets electronically (LaTeX, pdf or text file) by email. If you submit handwritten solutions, start each problem on a fresh page.
- Collaboration is encouraged, but all writeups must be done individually and must include names of all collaborators.
- Refering sources other than the text book and class notes is strongly discouraged. But if you do use an external source (eg., other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.
- The points for each problem are indicated on the side.
- Be clear in your writing.

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1. Prove that $NTIME(n) \neq L$. (15)

2. (a) **[Closure properties of NP]** (8)

   For a language L, we will abuse notation and say that $L(x) = 1$ iff $x \in L$ (i.e., $L = \{x|L(x) = 1\}$. Define the following Boolean functions on k bits (k is odd).

$$\begin{aligned}
\mathrm{OR}(x_1, x_2, \ldots, x_k) &= \bigvee_i x_i \\
\mathrm{AND}(x_1, x_2, \ldots, x_k) &= \bigwedge_i x_i \\
\oplus(x_1, x_2, \ldots, x_k) &= x_1 \oplus x_2 \oplus \cdots \oplus x_k \\
\mathrm{MAJ}(x_1, x_2, \ldots, x_k) &= \begin{cases} 1 & \text{more than } k/2 \text{ of the } x_i\text{'s are 1} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

   Given k languages $L_1, L_2, \ldots, L_k$ and a Boolean function f on k bits, define the language $f(L_1, L_2, \ldots, L_k)$ as follows

$$f(L_1, \ldots, L_k) = \{(x_1, x_2, \ldots, x_k)|f(L_1(x_1), L_2(x_2), \ldots, L_k(x_k)) = 1\}.$$

   If $L_1, L_2, \ldots, L_k$ are in NP, which of the following are in NP (assuming our current knowledge)? $\mathrm{OR}(L_1, \ldots, L_k), \mathrm{AND}(L_1, \ldots, L_k), \oplus(L_1, \ldots, L_k), \mathrm{MAJ}(L_1, \ldots, L_k)$. Explain with reasons. Answer the same question when NP is replaced by (a) P, (b) coNP and (c) NP $\cap$ coNP.

   [Optional] Identify the property of f, such that it satsifies

$$L_1, \ldots, L_k \in \mathrm{NP} \Rightarrow f(L_1, \ldots, L_k) \in \mathrm{NP}.$$

   (b) Identify the fallacy in the following argument. (8)

   Let $L \in \mathrm{NTIME}(n^3)$. Since $\mathrm{NTIME}(n^3) \subseteq \mathrm{NP}$, we have that $L \leq_p$ 3SAT. However, 3SAT $\in \mathrm{NTIME}(n)$. Hence, $L \in \mathrm{NTIME}(n)$. Thus, $\mathrm{NTIME}(n^3) \subseteq \mathrm{NTIME}(n)$ which implies the non-deterministic time-hierarchy is false.

3. **[Unique-witness]** (8)

   Show that U3COL $\in P^{SAT}$ where U3COL is defined as follows.

   $$\text{U3COL} = \{G = (V, E) | G \text{ has a unique 3-coloring (up to permutation of color labels)}\}.$$

4. **[MIN-FORMULA]** (5+6)

   Two Boolean formulae φ, φ' are said to be equivalent if they are defined on the same set of variables and compute the same Boolean function (i.e., $\varphi(x) = \varphi'(x), \forall x$). A Boolean formula is said to be minimal if no shorter formula is equivalent to it. Let MINFORMULA be the set of all minimal Boolean formulae.

   (a) Explain why this argument fails to show that MINFORMULA $\in coNP$: If $\varphi \notin$ MINFORMULA, then φ has a smaller equivalent formula. An NTM can verify that $\varphi \notin$ MINFORMULA by guessing that formula.

   (b) Prove that MINFORMULA $\in coNP^{SAT}$.

5. **[PSPACE problems]** (8+12)

   (a) Antakshari: Let Σ be a finite alphabet and let $S = \{s \mid s \in \Sigma^*\}$ be a set of strings over Σ. For $s = s_1 s_2 \ldots s_n \in \Sigma^*$, let suff($s$) and pre($s$) be s_1 and s_n, respectively.

   Let P_0 and P_1 be two players. The game of antakshari is played as follows: The game begins with P_0 choosing a string s_0 from S. The game then proceeds in rounds with player $P_{i \pmod 2}$ playing in round i as follows: During round i, player $P_{1 \pmod 2}$ picks a string s_i from S with the following property: pre(s_i) = suff(s_{i-1}), and for all $0 \leq j < i$, $s_i \neq s_j$, where s_j denotes the string picked in round j. A player is said to lose if she is not able to pick a string (i.e. no such string exists and she is stuck). Player P_0 is said to have a winning strategy on starting with s_0 if player P_0 wins the game irrespective of the subsequent moves of player P_1. Else, player P_1 is said to have a winning strategy.

   We define

   $$\text{Antakshari} = \{(S, s) \mid P_0 \text{ has a winning strategy on starting with } s\}$$

   Prove that Antakshari is in PSPACE.

   (b) Let $EQ_{REX} = \{\langle R, S\rangle | R \text{ and } S \text{ are equivalent regular expressions}\}$. Show that $EQ_{REX} \in PSPACE$.

   For extra credit, show that Antakshari (and EQ_{REX})) are in fact PSPACE-complete.

6. **[TQBF is hard for PSPACE under $\leq_m^{Log}$ reductions]** (7+8)

   Recall in class we proved that TQBF is hard for PSPACE under polynomial time many-one reductions. More precisely, given a $S(n)$-space machine M and input x, we produced in polynomial time a $O(S^2(n))$-sized quantified Boolean formula $\psi_{M,x}$ such that $\psi_{M,x}$ is true iff M accepts x in space $S(n)$.

   (a) Prove that the reduction can actually be performed in log-space. This implies that TQBF is hard for PSPACE even under logspace reductions

   (b) Use the above result to show that TQBF cannot be solved in space $O(n^{1/3})$.

7. **[well formed paranthesis]** (5+10)

   (a) Let A be the language of properly nested parentheses. For example, (()) and (()()) () are in A, but)(is not. Show that A is in L.

   (b) Let B be the language of properly nested parentheses and brackets. For example, ([()[]])([]) is in B but ([)] is not. Show that B is in L.