• Due Date: **6 Mar (Tue), 2018**

• Turn in your problem sets electronically (LaTeX, pdf or text file) by email. If you submit handwritten solutions, start each problem on a fresh page.

• Collaboration is encouraged, but all writeups must be done individually and must include names of all collaborators.

• Referring sources other than the text book and class notes is strongly discouraged. But if you do use an external source (e.g., other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.

• The points for each problem are indicated on the side.

• Be clear in your writing.

1. **[Boolean Formula Evaluation]** (10+8)

   (a) Prove that DFS of an undirected binary tree $T = (V, E)$ can be done in log-space.

   (b) A Boolean formula $\varphi$ on $n$ inputs is a directed tree with $n$ sources (vertices with no incoming edges) and one sink (vertex with no outgoing edges). All nonsource vertices are called gates and are labeled with one of $\lor, \land$ or $\neg$. The vertices labeled with $\lor$ or $\land$ have fan-in 2 and the vertices labeled with $\neg$ have fan-in 1. Let $x \in \{0, 1\}^n$ be some input. The output of $\varphi$ on $x$, denoted by $\varphi(x)$, is defined in the natural way. The Boolean formula evaluation problem deals with, given a formula $\varphi$ on $n$ inputs and $x \in \{0, 1\}^n$, computing the value of $\varphi(x)$. Show that formula evaluation can be done in logspace. More precisely, define

   $$FVAL = \{\langle \varphi, x \rangle | \varphi \text{ is a Boolean formula and } \varphi(x) = 1\}$$

   Prove that $FVAL \in L$.

   **Hint:** (a) You may assume that the tree is described as follows: For every vertex $v \in V$, there is a function $\text{next}_v : V \to V \cup \{\bot\}$ which gives a clockwise ordering of the edges around the vertex $v$. For every vertex $v$, there is a function $\text{next}_v(u)$ which gives the next neighbour in this cyclic ordering if $u$ is a neighbour of $v$ and $\bot$ otherwise. Finally, check that one can make this assumption without loss of generality.

2. **[Cycle in directed and undirected graphs]** (10+10)

   (a) Define UCYCLE $= \{\langle G \rangle | G$ is an undirected graph that contains a simple cycle $\}$. Show that UCYCLE $\in L$. Don’t use Reingold’s result that undirected reachability is in logspace.

   (b) Define CYCLE $= \{\langle G \rangle | G$ is an directed graph that contains a directed cycle $\}$. Show that CYCLE is NL-complete.
3. [Kannan’s Theorem: circuit lower bounds for PH] Problems 6.5 and 6.6. (10+10)

[Hint: Use Karp-Lipton-Sipser Theorem for part (a) and (q) to reduce to \( \Sigma^P_2 \).]

4. [circuit complexity of a threshold function] (7+10)

Consider the threshold function \( \text{Th}_{\geq 2}(x_1, \ldots, x_n) \), defined to be 1 iff at least two of the input variables are 1. For \( T \) a set of binary gates, define \( \text{size}_T(f) \) is the size of the smallest circuit that computes \( f \) using only gates from \( T \).

(a) Prove that \( \text{size}_{\{\land, \lor, \neg\}}(\text{Th}_2) \leq 4n + O(1) \). (Recall that our measure of circuit size includes the input variables.)

(b) Prove that \( \text{size}_{B_2}(\text{Th}_2) \geq 3n - O(1) \), where \( B_2 \) is the full binary basis (i.e., all Boolean functions on 2 inputs).

[Hint: Show that if two variables are inputs to some binary gate, then at least one of them must be used elsewhere in the circuit.]

5. \([\text{SUCCINCT-SET-COVER} \in \Sigma^P_2]\) Problem 5.11. (5)

6. [VC dimension] Problem 5.13. (10+10)

[Hint: Part (a): what is the size of the largest possible set \( X \) shattered by a collection of \( 2^m \) subsets? Part (b): Reduce from \( \exists \forall \exists \)-SAT, i.e., produce an instance \((C, k)\) from an instance \((\exists a \forall b \exists c \varphi(a, b, c), |a| = |b| = |c| = n)\) where \( \varphi \) is a formula in \( \exists \forall \exists \)-SAT.]

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