Problem Set 2

- Due Date: **28 Mar, 2019**
- Turn in your problem sets electronically (LaTeX, pdf or text file) by email. If you submit handwritten solutions, start each problem on a fresh page.
- Collaboration is encouraged, but all writeups must be done individually and must include names of all collaborators.
- Refering sources other than class notes and given references is discouraged. But if you do use an external source (eg., other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.
- The points for each problem are indicated on the side.
- Be clear in your writing.

1. **[EKR for p-biased Kneser Graph via Hoffman bound]**

   In this problem, we will generalize the Hoffman bound to weighted graphs and then use it to prove the Erdös-Ko-Rado theorem for the p-biased Kneser graph

   Let $G = (V, E, w)$ be a weighted graph where $w : V \to \mathbb{R}_{\geq 0}$. The size of the largest weighted independent set of G, denoted by $\alpha(G)$, is defined as follows:

   $$\alpha(G) := \max\{w(S) | S \subseteq V, S \text{ is an independent set}\},$$

   where $w(S) = \sum_{v \in S} w(v)$.

   For $p \in (0, 1)$, the p-biased Kneser graph, denoted by $\mathrm{Kn}_n^{(p)}$, is defined as follows.

   $$\begin{aligned} V &= 2^{[n]} = \{S \subset [n]\}, \\ E &= \{(S, T) \mid S \cap T = \emptyset\} \\ w_p(S) &= p^{|S|}(1-p)^{n-|S|} \end{aligned}$$

   (a) Let $p \in (0, 1/2)$. Show that $\alpha(\mathrm{Kn}_n^{(p)}) \geq p$. What happens when $p > 1/2$, $p = 1/2$?

   (b) Define the following weighted version of the Hoffman bound

   $$\theta_H(G) := \min_{\text{symmetric } B \in \mathbb{R}^{V \times V}, \lambda} \left(\frac{-\lambda}{1-\lambda} \right) \cdot w(V)$$

   subject to

   $$\begin{cases} B_{u,v} = 0 & \text{if } (u, v) \notin E \\ B\sqrt{w} = \sqrt{w} \\ B \succcurlyeq \lambda I \end{cases}$$

   where $\sqrt{w} \in \mathbb{R}^V$ denotes the vector given by $\sqrt{w}(v) = \sqrt{w(v)}$.

   Prove that $\alpha(G) \leq \theta_H(G)$.

(c) Using the weighted Hoffman bound, prove that $\alpha(\mathrm{Kn}_n^{(p)}) \leq p$.

[Hint: Find a suitable β for $n = 1$ case and tensorize]

2. **[Upper and lower bounds on the Shannon capacity of the 5-cycle]**

Given two (unweighted) graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, the *strong product* of G and H, denoted by $G \otimes H$ is the graph with vertex set $V_1 \times V_2$ and $((u_1, u_2), (v_1, v_2))$ is an edge if (i) $(u_1, u_2) \neq (v_1, v_2)$ and (ii) for $i = 1, 2$, either $u_i = v_i$ or $(u_i, v_i) \in E_i$. Given a graph G and $k \in \mathbb{Z}_{>0}$, the iterated strong product $G^{\otimes k}$ is defined as $G^{\otimes k} = G$ if $k = 1$ and $G^{\otimes (k-1)} \otimes G$ for larger k.

The Shannon capacity of a graph G, denoted by $\Sigma(G)$,is defined as

$$\Sigma(G) = \sup_k \sqrt[k]{\alpha(G^{\otimes k})} = \lim_{k \to \infty} \sqrt[k]{\alpha(G^{\otimes k})}.$$

The quantity was introduced by Shannon and is the best effective rate of communication per round with zero error on certain types of channels if we are allowed to use the channel for as many rounds as possible.

(a) Let C_5 denote the 5-cycle. Clearly, $\Sigma(C_5) \geq \alpha(C_5) = 2$. Show that this can be improved to $\Sigma(C_5) \geq \sqrt{5}$.

(b) Suppose $f : G \to \mathbb{R}$ is some function that satsifies $\alpha(G) \leq f(G)$ and $f(G \otimes H) \leq f(G) \cdot f(H)$. Show that $\Sigma(G) \leq f(G)$.

(c) Let $\bar{\chi}(G)$ denote the minimum number of cliques required to cover the edge set of G or equivalently $\bar{\chi}(G) = \chi(\bar{G})$, the chromatic number of $\bar{G}$ (the minimum number of colours required to colour the vertices of G such that every edge in $\bar{G}$ is non-monochhromatic).

Using the previous part or otherwise, show that $\Sigma(G) \leq \bar{\chi}(G)$. Conclude that $\Sigma(G) \leq 3$.

3. **[Orthonormal representations of a Graph]**

Let $G = ([n], E)$ be a graph on the vertex set $[n] = 1, 2, \ldots, n$. An *orthonormal representation* of G is a system $(v_1, v_2, \ldots, v_n)$ of unit vectors in a Euclidean space such that if i and j are non-adjacent vectors, then v_i and v_j is orthtogonal.

(a) Prove that if $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$ are orthonormal representations of $G = ([n], E)$ and $H = ([m], F)$ respectively, then the system $(u_i \otimes v_j)_{i,j}$ is an orthormal representation of $G \otimes H$.

The value of an orthonormal representation $(u_1, \ldots, u_n)$ is defined to be

$$\min_c \max_{i \in [n]} \frac{1}{\langle c, u_i \rangle^2},$$

where c ranges over all unit vectors. The vector c yielding the minimum is called the handle of the representation. Let $\vartheta(G)$ denote the minimum value over all representations of G.

(b) Prove that $\vartheta(G \otimes H) \leq \vartheta(G) \cdot \vartheta(H)$.

(c) Prove that $\alpha(G) \leq \vartheta(G)$.

(d) Conclude that $\Sigma(G) \leq \vartheta(G)$.

(e) Prove that $\vartheta(C_5) \leq \sqrt{5}$.

[Hint: Consider the orthonormal representation given by a 5-pronged umbrella opened up such that non-adjacent spokes are orthogonal.]

(f) Conclude that $\Sigma(C_5) = \vartheta(C_5) = \sqrt{5}$.

4. **[Lovász Theta function via orthogonal representations]**

In class, we defined the Lovász Theta function of a graph as follows

$$\theta_L(G) = \min_{\text{symmetric } M \in \mathbb{R}^{V \times V}, \lambda} \lambda$$

$$\text{subject to}$$

$$\begin{cases} M_{u,v} = 1 & \text{if } u = v \text{ or } (u,v) \notin E \\ M \preccurlyeq \lambda I \end{cases}$$

In this problem, we will show that $\theta_L(G) = \vartheta(G)$.

(a) Given an orthonormal representation $(u_1, \ldots, u_n)$ and a handle c, define the vectors $v_i := c - u_i / \langle c, u_i \rangle$. Consider the matrix $N = (\langle v_i, v_j \rangle)_{i,j}$ and $D = diag(\vartheta(G) - 1/\langle c, u_i \rangle^2)$ and $M = N + D$. Use matrix M to show that $\theta_L(G) \leq \vartheta(G)$.

(b) Let (M, λ) be a feasible solution to the SDP formulation of $\theta_L(G)$. Since $\lambda I - M \succcurlyeq 0$, we have that there exist vectors $v_1, \ldots, v_n$ such that $\lambda I - M = (\langle v_i, v_j \rangle)_{i,j}$. Let c be any unit vector orthogonal to all the v_i's. Define $u_i := (c + v_i)/\sqrt{\lambda}$. Show that $(u_1, \ldots, u_n)$ is an orthonormal representation of G. Using this prove that $\vartheta(G) \leq \lambda$. Hence, conclude that $\vartheta(G) \leq \theta_L(G)$.