Today

#P-completeness

Valiant's Theorem:

Permanent is #P-complete

Downward Self-Reducibility

Toda's Theorem: $PH \subseteq P^{\#P}$

Lecture 17: Computational Complexity

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Recall

#P-completeness:

$\cdot\ f, g \in \#P, \quad f \leq_p g$

One attempt: $f \in FP^g$ (relaxed defn)

Two attempt: (stricter defn)

$f(x) = g(y)$

(#SAT is #P-complete (even under this stricter defn)

Use the more relaxed defn

Permanent is #P-complete (under relaxed defn)

Valiant's Theorem: Permanent of $\{0,1\}$-matrices is #P-complete.

Pf: Step 1: Permanent of $\{0,1,-1\}$-matrices is #P-complete

Reduce from #SAT

$\varphi \longmapsto G_\varphi$ (weights $0, 1, -1$)

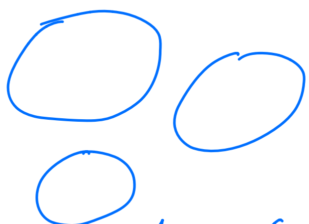
$(-2)^{3m/2} \cdot \#SAT(\varphi) = \cdot \text{Cycle-Cover}(G_\varphi)$

$m = \#\text{clauses of } \varphi$

$\text{Perm}(\text{Adj}(G_\varphi))$

$\mathcal{C}$- Cycle Cover

G:



$val(\mathcal{C}) = \prod_{e \in \mathcal{C}} e$

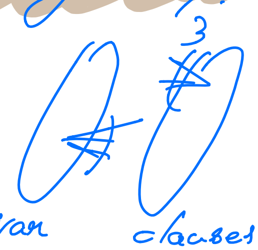
Total cycle(G)

$= \sum_{\mathcal{C}\text{-cycle cover}} val(\mathcal{C})$

(covers all vertices
vertex-disjoint cycles)
specified by σ

Assumption: Every variable appears in as many literals positively as negatively.

add $(x \vee \bar{x} \vee \bar{x})$ or $(x \vee x \vee \bar{x})$



$\sum \text{even} = 3m$

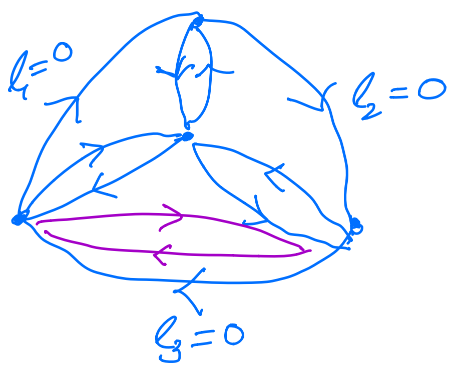
Redn 3 ingredients

① Variable Gadget

x

2. Clause Gadget

$C = (l_1 \vee l_2 \vee l_3)$: l_i - literals.



For every satisfying assignment to C there is a cycle cover of val 1, that involves exactly those literals that are falsified.

There is no cycle-cover involving the 3 outer edges.

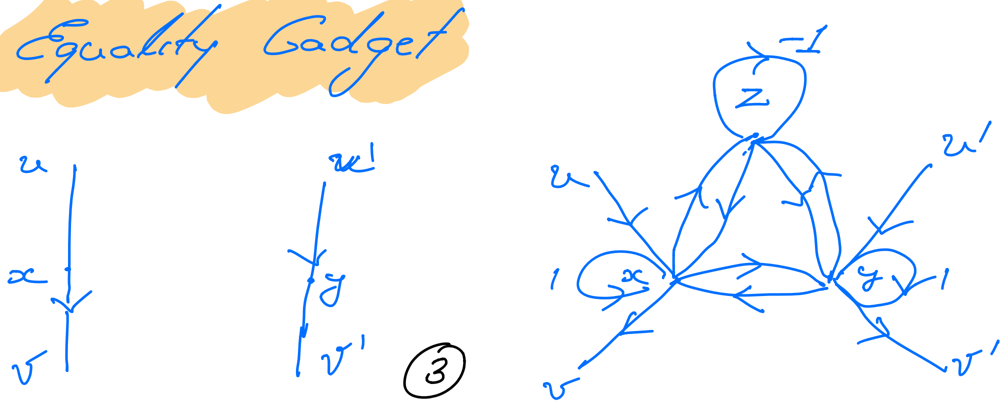
And these are all the cycle covers.

Consistent Cycle Cover: One in which the clause & variable gadgets are consistent

$$\#\text{Consistent Cycle Covers} = \#SAT(\varphi)$$

(There could be inconsistent cycle covers)

3 Equality Gadget



uv & $u'v'$ are both taken

$\rightarrow$ Cycle cover with val -1

(that involves self-loop 2)

uv & $u'v'$ are both not taken

$\rightarrow$ 2 Cycle covers w/ val 1 each

($x \to y \to z \to x$ or $x \to z \to y \to x$)

$$\#\text{Consistent Cycle Covers} = (-1)^p (2)^q \, \#SAT(\varphi)$$

$p = \#$ positive literals

$q = \#$ negative literals

$$= (-2)^{3m/2} \cdot \#SAT(\varphi)$$

For every inconsistent setting there is an involution mapping cycle covers of val ω to val $-\omega$

$$\sum_{C\text{-Cycle Cover}} val(C) = \sum_{C\text{-cons Cycle}} val(C) + \sum_{C\text{-incon}} val(C)$$

$$= (-2)^{3m/2} \cdot \#SAT(\varphi) + 0.$$

$$= (-2)^{3m/2} \cdot \#SAT(\varphi).$$

Completes Step 1.

Step 2: Reduce perm of integer-valued matrices

$\downarrow$

perm of non-negative-integer valued matrices.

$M = \max_{ij} \{|M_{ij}|\}$

$\mathrm{perm}(M) \leq 2\, n!\, M^n =: Q$. [n - given in unary, $n = \dim(M)$]

M' : $M'_{ij} = Q + M_{ij}$ $\quad$ M' - has non-negative entries.

$\mathrm{perm}(M) = \mathrm{perm}(M') \pmod{Q}$

Step 3: Reduce permanent of non-negative integer valued matrices

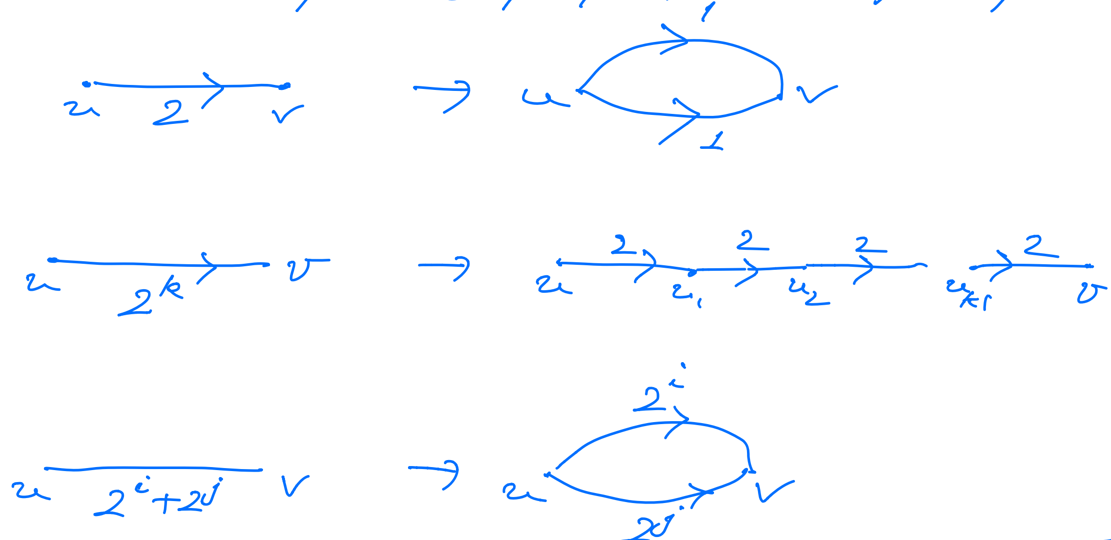
$\downarrow$

permanent of $\{0,1\}$- integer valued matrices

Use the char of perm as the total cycle cover of a graph.

G (w/ non-negative weights) $\longmapsto$ G' (w/ $\{0,1\}$-weights)

$\mathrm{perm}(\mathrm{Adj}(G)) = \mathrm{perm}(\mathrm{Adj}(G'))$



Conclusion: Permanent is #P-complete.
(for even {0,1}-valued matrices).

Permanent:

1. Downward Self-reducibility

$$\mathrm{per}(M) = \sum m_{1i} \cdot \mathrm{perm}(M_{1i})$$

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ & & & \end{bmatrix}$$

M_{1i} – matrix formed by deleting 1^{st} row & i^{th} column

$$\mathrm{Perm}_{n\times n} \in FP^{\mathrm{Perm}_{(n-1)\times(n-1)}}$$

2. Random Self Reducibility

Permanent is hard on worst-case
$\Downarrow$
Permanent is hard on average.

Toda's Theorem:

How powerful is #P?

Easiness: $\#P \in FPSPACE$.
(i.e. $P^{\#P} \subseteq PSPACE$)

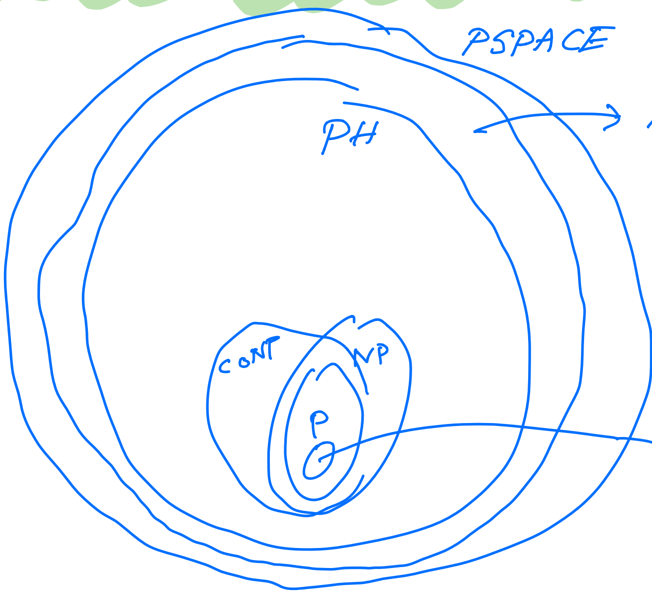
Hardness: $NP \subseteq P^{\#P}$

What about PH?

Observe: $\#P = FP \Rightarrow P = NP \Rightarrow P = PH$

(this is not known to imply $PH \subseteq P^{\#P}$)

Toda's Theorem: $PH \subseteq P^{\#P}$



$P^{\#P} \ni$ Permanent

VP vs VNP qn

$NC \ni$ determinant

Toda's Theorem

Part 1: $PH \subseteq BP\cdot\oplus P$

Part 2: $BP\cdot\oplus P \subseteq P^{\oplus P} \subseteq P^{\#P}$

} Brings in randomized classes though original stmt is "randomness-free"

Part 1: $PH \subseteq BP\cdot\oplus P$

Today: Simpler: $NP \subseteq BP\cdot\oplus P$)

$\oplus P$: ??

$\#P$ – $\#$ acc paths

NP – $\exists$ acc path

PP – m.s.b $\#$ acc paths

$\oplus P$ – l.s.b $\#$ acc paths

$\oplus P$: $L \in \oplus P$ if $\exists$ a poly time TM M & poly p s.t

$$x \in L \iff \#\{y \in \{0,1\}^{p(|x|)} \mid M(x,y)=1\} \text{ is odd.}$$

eg: $\oplus SAT = \{\varphi \mid \varphi$ has an odd $\#$ of satisfying assignments$\}$

$\oplus SAT \in \oplus P$.

Valiant – Vazirani: Randomized redn from SAT to USAT

$$\varphi \longmapsto \psi := \varphi \wedge h$$

$$\varphi \in SAT \Rightarrow \Pr_h[\varphi \wedge h \in USAT_Y] \geq \frac{1}{8n}$$

$$\varphi \notin SAT \Rightarrow \Pr_h[\varphi \wedge h \in USAT_N] = 1$$

$$\varphi \in SAT \Rightarrow \Pr_h[\psi \in \oplus SAT] \geq \frac{1}{8n}.$$

$$\varphi \notin SAT \Rightarrow \Pr_h[\psi \notin \oplus SAT] = 1.$$

Want is the following: $\forall m$

$$\left.\begin{aligned} \varphi \in SAT &\Rightarrow \Pr_R[\psi \in \oplus SAT] \geq 1 - \frac{1}{2^m} \\ \varphi \notin SAT &\Rightarrow \Pr_R[\psi \notin \oplus SAT] = 1. \end{aligned}\right\}$$

(Prove $NP \subseteq BP\cdot\oplus P$)