Linear programming

Zero-sum two player game:- Two players: Row and Column.

Row plays a move r
Column plays a move c.

↳ In this situation Row receives $A(r,c)$
Column receives $\underbrace{-A(r,c)}_{A:\ \text{payoff matrix}}$.

Q: Given the payoff matrix $\overbrace{A}^{\text{finite}}$: let us start with the question: "What is the maximum number of points that player Row can win?"

– (Does this question make sense?)

If the column player were 'collaborating' [trying to make the row player win as much as possible, not worrying by their own loss] then this is just $\max_{r,c} A(r,c)$

We want to work in the setting where both the row and column players want to maximize their own payoff.

– (This is the setting that is useful for applications of this framework.)

One response

$$\max_{r} \min_{c} A(r,c). \quad — \quad (1)$$

"Row player playing first"

Why not : $\min_c \max_r A(r,c)$? — (2)

"Coloum player playing first"

Are they always equal?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Option (1) :- 3 = Option (2) :- 3

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

Option (1) :- 1 < Option (2) :- (2) ✓

Can we have an example where Option (1) > Option (2) ?

Suggestion :- $\begin{pmatrix} -1 & -2 \\ -2 & 0 \end{pmatrix}$

Option (1) :- −2 < Option (2) :- (−1) (*)

Theorem :- (*) For any $m \times n$ matrix A

$$\max_{i \in [m]} \min_{j \in [n]} A_{ij} \leq \min_{j \in [n]} \max_{i \in [m]} A_{ij}$$

"Moving second is at least as good."

Proof :- $\min_{j \in [n]} A_{ij} \leq A_{ik} \quad \forall i, k.$

Maximize both sides over i, for every fixed k.

This gives:

$$\underbrace{\max_{i \in [m]} \min_{j \in [n]} A_{ij}}_{\text{Does not depend upon } k.} \leq \underbrace{\max_{i \in [m]} A_{ik}}_{\text{depends only on } k}, \forall k.$$

Since this is true for all k we can take the k that minimizes the left hand side to get:

$$\max_{i \in [m]} \min_{j \in [n]} A_{ij} \leq \min_{j \in [n]} \max_{i \in [m]} A_{ij}.$$

(Above setting is referred to as the 'pure strategy' setting.)

A slightly different setting:

Now instead of publishing a single strategy, each player can publish a probability distribution over all their previously possible moves.

→ Then a 'pure' move is sampled independently (for each player) from their published distribution, and the corresponding 'pure' payoff is the 'payoff' for this realization.

– The 'mixed' payoff is computed as the <u>expected</u> payoff over the randomness of the sampling of the moves. The

Row player publishes : $\vec{r} \in \Delta_{rows}$ (probability distributions over the set of rows)

Column player publishes : $\vec{c} \in \Delta_{\text{columns}}$ (probability distributions over the set of columns)

Expected payoff: $\mathbb{E}_{\substack{i \sim \vec{r} \\ j \sim \vec{c}}} [A(i,j)]$

$$= \sum_i \sum_j r(i) A(i,j) c(j)$$

(Because of independence)

$$= r^T A c.$$

(Thinking of $\vec{r}$ and $\vec{c}$ as column vectors)

Again: the players want to <u>maximize</u> their own expected payoff (equivalently, <u>minimize</u> the opponent's payoff).

So, if the row player publishes <u>first</u>, the row player's expected payoff is:

$$\max_{\vec{r} \in \Delta_{\text{row}}} \min_{\vec{c} \in \Delta_{\text{column}}} \vec{r}^T A \vec{c}$$

If the column player publishes <u>first</u>, the row player's expected payoff is:

$$\min_{\vec{c} \in \Delta_{\text{column}}} \max_{\vec{r} \in \Delta_{\text{row}}} \vec{r}^T A \vec{c}$$

Essentially the same argument (assuming there are only finitely many pure strategies per player) shows that

$$\boxed{\max_{\vec{r}\in\Delta_{row}} \min_{\vec{c}\in\Delta_{column}} \vec{r}^T A \vec{c} \leq \min_{\vec{c}\in\Delta_{column}} \max_{\vec{r}\in\Delta_{row}} \vec{r}^T A \vec{c}}$$

Von-Neumann min-max theorem :- (Assume both players have finitely many strategies) Then,

$$\max_{\vec{r}\in\Delta_{row}} \min_{\vec{c}\in\Delta_{column}} \vec{r}^T A \vec{c} = \min_{\vec{c}\in\Delta_{column}} \max_{\vec{r}\in\Delta_{row}} \vec{r}^T A \vec{c}.$$

Linear programming duality : "Standard form"

$$\min \ c^T x \leftarrow \text{Objective fn.}$$

$$\underbrace{\text{Constraints}} \begin{cases} Ax = b \\ x \geq 0, \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m\times n} \\ \text{(Component wise)} \end{cases}$$

If there is no x satisfying the constraints then we say the program is infeasible.

An $x \in \mathbb{R}^n$ satisfying the constraints is called feasible.

Otherwise :- Either : $\forall\ r \in \mathbb{R}, \ \exists$ a feasible x s.t. $c^T x < r$

— Program is said to be unbounded (below).

Or :- There is a finite p^* s.t. there is a feasible x with $c^T x = p^*$, and s.t. there

Convention:- $\max \{\} = -\infty$, $\min \{\} = \infty$.

are no feasible x with $c^T x < p^*$.

$\rightarrow$ p^* is called the optimal value of the program.

Dual program :- Suppose we take some arbitrary linear combination of the constraints $Ax = b$. That means $y \in \mathbb{R}^m$, and the combination gives.

$$(y^T A) x = y^T b \qquad (x \text{ feasible, but so far } x \geq 0 \text{ has not been used})$$

$$\equiv \quad (A^T y)^T x = y^T b = b^T y$$

So, if $\vec{y}$ is such that

$$(A^T y) \leq c \quad \text{component wise.}$$

then, $\forall\, x \in \mathbb{R}^n$ s.t. $x \geq 0$, $Ax = b$ (i.e. for every feasible x) we have.

$$b^T y = (A^T y)^T x \leq c^T x$$

So, $\forall y$ s.t.

$$A^T y \leq c$$

we have

$$c^T x \geq b^T y \quad \forall \text{ feasible } x.$$

So this shows that if p^* exists then

$$p^* \geq d^* := \left.\begin{aligned} \max\ & b^T y \\ & A^T y \leq c. \\ & y \in \mathbb{R}^m. \end{aligned}\right\} \text{Dual Program}$$

Original program is called the primal program.

Next time:- Duality theorem:- When both primal and dual are feasible, $p^* = d^*$. Only other possibilites are:-

(1) Primal and dual both infeasible

(2) Primal infeasible and dual unbounded above. ("$d^* = +\infty$")

(3) Dual infeasible and primal unbounded below. ("$p^* = -\infty$")