Lecture 9 :- Rounding for the MAX-CUT SDP.

We have $\{v_i\}_{i=1}^{n} \subseteq \mathbb{R}^n$ s.t. $\|v_i\| = 1$ $1 \le i \le n$, SVD.

$$\text{MAX-CUT} \le M = \frac{1}{2} \sum_{\substack{e \in E \\ e = \{i,j\}}} w_e (1 - v_i^T v_j).$$

Goemans-Williamson Rounding :- Pick a unit vector $u \in \mathbb{S}^{n-1}$ (the unit sphere in $\mathbb{R}^n$) uniformly at random

[Q: How do we sample such a u? What properties does such a u have?]

Rounding process :- Let H_u be the hyperplane through the origin normal to u. The 'cut' is the set of all vectors which lie on one side of H_u. Formally:

$$S := \{ i \in V \mid u^T v_i > 0 \}.$$

Q: What is $\mathbb{E}_{u \in \mathbb{S}^{n-1}} [\text{Cut}(S)]$?

$$\text{Cut}(S) = \sum_{\substack{e \in E \\ e = \{i,j\}}} w_e \, \mathbb{I}[|e \cap S| = 1]$$

$$= \sum_{\substack{e \in E \\ e = \{i,j\}}} w_e \, \mathbb{I}[\text{Exactly one of } u^T v_i, u^T v_j \text{ is positive}]$$

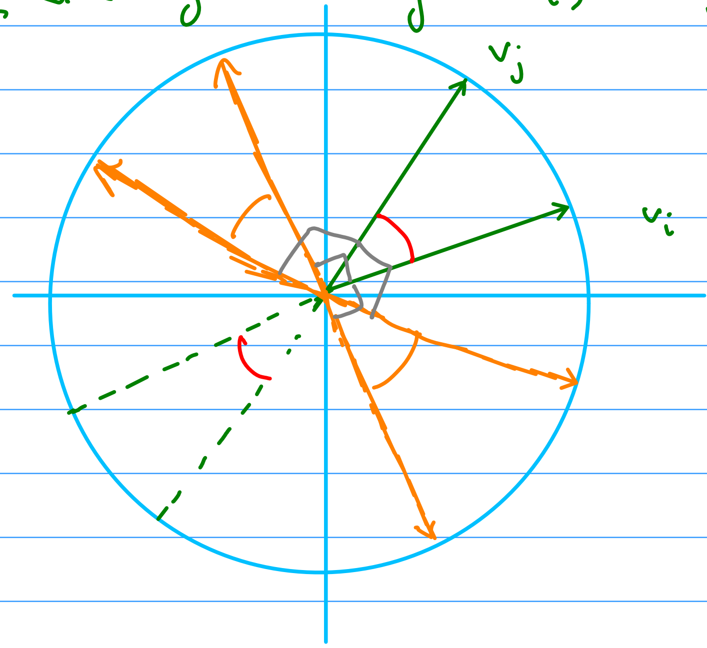
$$\mathbb{E}_{u \in \mathbb{S}^{n-1}} [\text{Cut}(S)] = \sum_{\substack{e \in E \\ e = \{i,j\}}} w_e \cdot \Pr[\text{Exactly one of } u^T v_i, u^T v_j \text{ is positive}]$$

So, all we need to compute is (assume $v_i \neq v_j$)

$$\Pr\left[\text{Exactly one of } u^T v_i,\ u^T v_j \text{ is positive}\right].$$

In 2D:-

(v_i, v_j, u are all in 2 dimensions)



Orange: u s.t. exactly one of $u^T v_i$, $u^T v_j$ is positive.

Red: line perp to separates v_i and v_j

So if the angle between v_i and v_j is Θ (which direction is the angle measured?), the allowed 'angle' for u is 2Θ.

$$\text{So } \Pr\left[\begin{array}{c}\text{Exactly one of } u^T v_i,\ u^T v_j \text{ is} \\ \text{positive}\end{array}\right] = \frac{2\Theta}{2\pi} = \frac{\Theta}{\pi}.$$

[In 2 dimensions!!]

Also, this argument as long as Θ is chosen to be in $(0, \pi]$.

n- dimensions

Observation:- Let V be the two dimensional spanned by v_i & v_j. Then the distribution of $\dfrac{\mathrm{Proj}_V(u)}{\|\mathrm{Proj}_V(u)\|}$ (where $\mathrm{Proj}_V(u)$ is the orthogonal projection of u onto V) is uniform on the set of unit vectors in V, conditional on $\mathrm{Proj}_V(u) \neq 0$, when $u \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.

Since $u^T w = \mathrm{Proj}_V(u)^T w$ for any $w \in V$, we therefore get from the above two-dimensional argument that, even in dimensions: (assuming $v_i \neq v_j$)

$$\Pr[\text{Exactly one of } u^T v_i,\ u^T v_j \text{ is positive}] = \frac{\theta_{ij}}{\pi},$$

where $\theta_{ij} \in (0, \pi]$ is the angle between v_i and v_j. Thus we get.

$$E[\mathrm{Cut}(S)] = \sum_{\substack{e \in E \\ e=\{i,j\}}} w_e \frac{\theta_{ij}}{\pi}.$$

On the other hand, we also had.

$$\text{MAX-CUT} \leq M = \sum_{\substack{e \in E \\ e=\{i,j\}}} w_e \frac{(1-\cos\theta_{ij})}{2}$$

$$\because \quad v_i^T v_j = \cos\theta_{ij}.$$

We only use those edges here for which $v_i \neq v_j$. Edges with $v_i = v_j$ contribute 0 to both sums

$$\alpha := \inf_{\theta_{ij} \in (0,\pi]} \frac{(\theta_{ij}/\pi)}{\left(\frac{1-\cos\theta_{ij}}{2}\right)}$$

$$E[\mathrm{Cut}(S)] = \sum_{e \in E} w_e \left(\frac{\theta_{ij}}{\pi}\right)$$

$$\geq \alpha \sum_{\substack{e \in E \\ e=\{i,j\}}} w_e \left(\frac{1-\cos\theta_{ij}}{2}\right)$$

$= \alpha M \geq \alpha$ MAX-CUT.

So, this rounding algorithm generates a cut S s.t.

$$\text{MAX-CUT} \geq E[\text{cut}(S)] \geq \alpha M \geq \alpha \text{ MAX-CUT}.$$

$$\alpha \approx 0.878.$$

<u>Questions</u>: (1) Derandomize ?? ((Can be done by method of conditional expectations) (Mahajan-Ramesh)

↳ Compute conditional expectations of Gaussians.

(2) Is α tight ??

- 'Integrality gap' examples are known [Feige - Schechtmann]
  - A sequence of Graphs $(G_n)_{n \geq n_0}$ s.t.

    $\text{MAX-CUT} \leq (\alpha + o(1)) M$ for the graph.

- Khot, Raghavendra and many other works: the conjecture is that this kind of SDP-based algorithm might be the 'optimal' algorithm for a large class of CSPs. The conjecture follows if one assumes the Unique Games Conjecture of Khot.

  ↳ MAX-CUT: Khot-Kindler-Mossel-O'Donnell

  Large class of CSPs - Raghavendra]

For vertex cover and related CSPs ←
- Khot-Regev
- Kumar-Manokaran-Tulsiani-Vishnoi