## Today

Spectral Methods (Part VII)

- Expander Graphs & Applications

CSS.205.1
Toolkit in TCS
- Lecture #26
(25 May '21)
Instructor: Prahladh Harsha

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## Recall:

### Spectral Expansion

A graph $G=(V,E)$ on $n$-nodes & $d$-regular is called a spectral expander w/ spectral expansion $\gamma = 1-\omega$ if

$\omega \geq \max\{\omega_2, |\omega_n|\}$

where $\omega_i$ - are the evalues of the r.walk matrix induced by $G$.

$\gamma$ - spectral gap ($G$ is $[n,d,\gamma]$-expander)

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### Vertex Expander

An $n$-node $d$-regular graph $G=(V,E)$ is a $(K,A)$-expander if $\forall S \subseteq V$,

$|S| \leq k \Rightarrow |N(S)| \geq A|S|$.

$N_+(S) = N(S) \cup S$

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**Theorem** [Spectral Expander $\Rightarrow$ Vertex Expansion]

$G$ is $(n, d, \gamma)$-spectral expander
then for all $\alpha \in (0,1)$
$G$ is $\left(\alpha n, \dfrac{1}{\omega^2(1-\alpha)+\alpha}\right)$-vertex expander
where $\omega = 1-\gamma$.

<u>Cor.</u>: $\omega < 1 \Rightarrow G$ is $\left(\frac{n}{2}, 1+\delta\right)$-expander
for some $\delta \in (0,1)$.

<u>Qn:</u>
Do expanders exist?

Can we construct them explicitly?

— (Random graphs are very good vertex expanders)

<u>Thm</u>: $\forall$ constants $d$, $\exists\, \alpha > 0$
random $d$-regular graph on $n$-nodes
is a $(\alpha n, d - 1.01)$-vertex expander
w.h.p

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What about spectral expansion?

Thm [Friedman]: $\forall d$, random $d$-regular graph on $n$ nodes satisfies

$$\omega \leq \frac{2\sqrt{d-1}}{d} + o_n(1) \quad \text{w/ prob} \quad 1 - o_n(1)$$

where $o_n(1) \to 0$ as $n \to \infty$.

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Can we do better than the above random construction?

YES, for instance a complete graph on $d$ vertices w/ self loops

Thm [Alon-Boppana]
For any family of $d$-regular graphs

$$\omega(G) \geq \frac{2\sqrt{d-1}}{d} - o_n(1) \quad \text{where}$$

$o_n(1) \to 0$ as $n \to \infty$.

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Explicit Constructions:

- mildly explicit – poly$(|G|)$
- fully explicit – there is a polylog$n$ time algo that given $i \in [n]$, $a \in [d]$ outputs the $a^{th}$ neighbour of $i$

(1) $m$ – positive integer

$V = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$

$(x,y) \to (x,y), (x+1,y), (x,y+1), (x,x+y), (-y,x)$
$(x-1,y), (x,y-1), (x,y-x), (y,-x)$

Margulis

$G$ is $(m^2, 8, \gamma)$ - spectral expander
for some $\gamma \in (0,1)$

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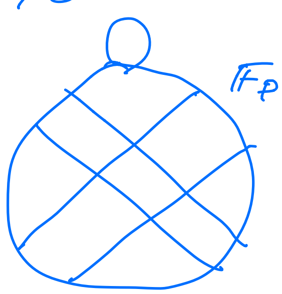
(2) $V = \mathbb{Z}/p\mathbb{Z}$ $p$ - prime

3 - regular.

$x \mapsto x+1, \quad x-1, \quad 1/x$

$1/0 \to 0$



$(p, 3, \gamma)$ - spectral expander
for some $\gamma \in (0,1)$

- 3/16 - Selberg Theorem

(3) Lubotzky - Philips - Sarnak [LPS]
- Ramanujan graphs.

$p$ - prime
$q = p^k$ $\qquad q \equiv 1 \pmod 4$
$\exists\, i \in \mathbb{F}_q, \quad i^2 \equiv -1 \pmod q$

$V = \mathbb{F}_q = \mathbb{F}_{p^k}$

$$z \in \mathbb{F}_q \longmapsto \frac{(a_0 + i a_1) z + (a_2 + i a_3)}{(-a_2 + i a_3) z + (a_0 - i a_1)}$$

for $(a_0, a_1, a_2, a_3)$ satisfying

$a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$

$a_0$ - odd, positive

$a_1, a_2, a_3$ - even.

deg of LPS-graph $= p+1$

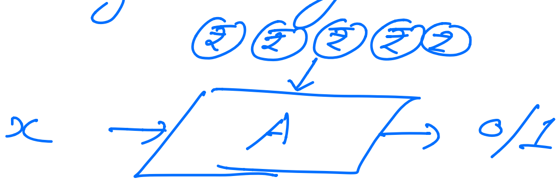
$$\omega(LPS) \leq \frac{2\sqrt{d-1}}{d} = \frac{2\sqrt{p}}{p+1}.$$

Ramanujan graphs: Family of expanders $d$-regular s.t. $\omega(G) \leq \frac{2\sqrt{d-1}}{d}$

## Application of Expanders

Derandomization

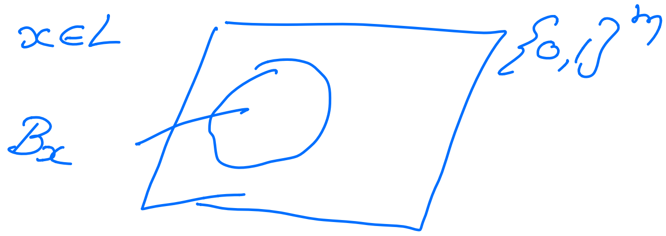
Randomized Algorithms. (RP)



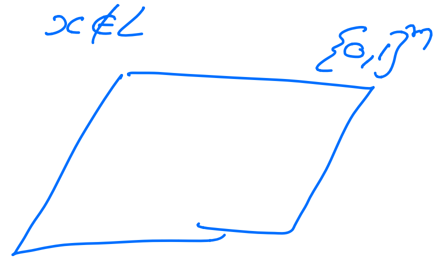

Prime / Composite:

$x$ - composite $\Rightarrow \Pr_r[A(x,r) = \text{comp}] \geq 1/2$

$x$ - prime $\Rightarrow \Pr_r[A(x,r) = \text{prime}] = 1$




space of random coins.




$B_x = \phi$.

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## Error Reduction

$\frac{1}{2} \rightarrow \delta$ by repeating independent runs of the Alg $A$ $O(\log(\frac{1}{\delta}))$ times.

$A(x, r_1), \ldots \quad A(x, r_k)$. $k = O(\log \frac{1}{\delta})$

$\frac{1}{2} \longmapsto \delta$.

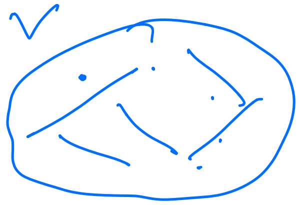
$\#$ random coins $= O(\log(\frac{1}{\delta})) \cdot m = km$

<u>Qn</u>: How many additional random coins are needed to reduce error from $1/2$ to $\delta$?

$\{0,1\}^m \longleftrightarrow V$

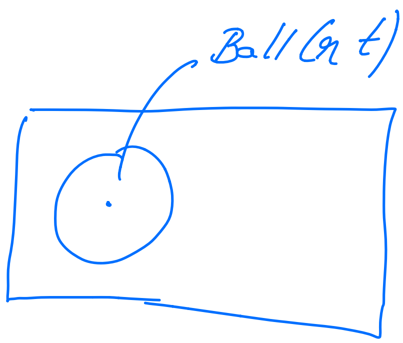
vertex set of a graph

$V$



Impose an $d$-regular expander $G$ on $V$ s.t $G$ is $(\frac{n}{2}, A)$-vertex expander for some $A > 1$. (independent of $n$).

$n = \{0,1\}^m$




$\underline{A'}$

1. Pick a random vertex $x \in \{0,1\}^n$
2. Choose all vertices $x_1 \dots x_k$ within distance $t$ of $x$
3. Run $A$ for all $x' \in Ball(x,t)$ & accept if any of them accept.

<u>Remarks:</u>

If $t = O(\log n)$
$k = poly(n)$
$\rightarrow$ alg is polytime alg.

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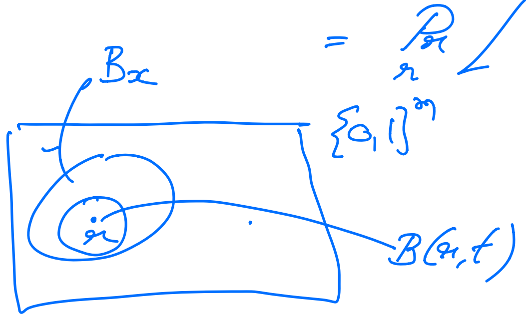
Error of algorithm

$x \notin L \Rightarrow \Pr[A' = \text{correct}] = 1$

$x \in L \Rightarrow \Pr_x[A' \text{ is wrong}]$

$= \Pr_x[Ball(x,t) \subseteq B_x]$




$Ball(x,t) = N_+^{(t)}(x)$

$\supseteq N^{(t)}(x)$

$$S_0 = \{r\} \quad |N(S_0)| \geq A \cdot 1 \quad |N(N(S_0))| \geq A^2$$

$$\ldots \quad |N^{(t)}(S)| \geq A^t$$

$$Ball(r,t) \supseteq N^{(t)}(S)$$

$$2^m \tfrac{1}{2} \geq |Ball(r,t)| \geq |N^{(t)}(S)| \geq A^t$$

Let $BAD(x) = \{r \mid B(r,t) \subseteq B_x\}$.

$$\Pr_r[A'\text{-wrong}] = \Pr_r[r \in BAD(x)]$$

$$= \frac{BAD(x)}{2^m} = p$$

$$2^{m-1} \geq |N^{(t)}(BAD(x))| \geq A^t \cdot |BAD(x)|$$

$$\geq A^t \cdot p \cdot 2^m$$

$$p \leq \frac{1}{2A^t} \qquad t = O(\log n)$$

$$= \frac{1}{poly(n)}$$

Thm. $t = O(\log n)$: $\Pr_r[BALL(r,t) \subseteq B_x] \leq \frac{1}{poly(n)}$.

[Karp-Pippenger-Sipser].

Next lecture

$\frac{1}{2} \longmapsto \delta$ : #random coins

$\exp(-k)$ $= m \cdot O(\log \frac{1}{\delta})$ [Name]

$= O(mk)$

Use expanders (spectral)

#random coins

$= m + O(\log \frac{1}{\delta})$

$= m + O(k)$