Today

Lovasz Theta Function

CSS.205.1
Toolkit in TCS
- Lecture #27
(26 May '21)
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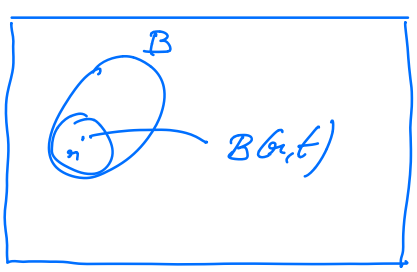
Recap: (derandomization using expanders)

Thm [Karp-Pippenger-Sipser]
Suppose $\exists\, A > 1$ st if G - an n-vertex d-regular graph is an $\left(\frac{n}{2r}, A\right)$-vertex expander then for any $B \subseteq V$, $|B| \le N/2$.

$$\Pr_{r \leftarrow V}\left[B(r,t) \subseteq B\right] \le \frac{1}{2 \cdot A^t}$$



$V =$

Application.
RP error reduction.

(1) Error $\frac{1}{2} \to \delta$

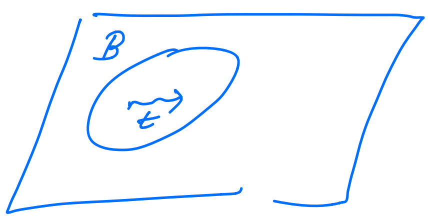
Set $\frac{1}{2A^t} \le \delta$;

$$t = \left\lceil \frac{\log\left(\frac{1}{\delta}\right) - 1}{\log A} \right\rceil$$

(2) $|B(x,t)| \le (d+1)^t$ – polynomial as long as $t = O(\log n)$

Hence, $\delta = \frac{1}{\mathrm{poly}(n)}$ (ie, any inverse polynomial) is achievable by above method.

Qn: What about smaller δ?
(possibly at the cost of a few random bits).



Replace ball of radius t w/ walk of radius t.

Hitting Set Lemma for RW on expanders (Ajtai - Komlos - Szemeredi)

W - reversible random walk ; $\omega = \max\{\omega_2, |\omega_n|\}$

$B \subseteq V$; $\pi(B) \triangleq \mu$

$X_1 \ldots X_t$ - random walk of length t starting at $X_1 \sim \pi$.

$$(*) = \Pr_{X_1 \ldots X_t}\Big[\bigwedge_{i \in [t]} (X_i \in B)\Big] \le \mu\big(\mu + \omega(1-\mu)\big)^{t-1}$$

Remark: $\mu = 1/2 \qquad \omega < 1 \qquad \mu + \omega(1-\mu) = \frac{1+\omega}{2}$

$$\frac{1}{2^t}(1+\omega)^{t-1} \doteq \exp(-t).$$

$\#$random coins $= \log n + (t-1)\log d$

$([n, d, 1-\omega]$-expander$)$ $= \log n + O(t)$

(in contrast to independently $O(t \cdot \log n)$.

Proof of Hitting Set Lemma:

$$(*) = \sum_{i_1, i_2 \ldots i_t \in B} \Pr[X_1 = i_1 \wedge X_2 = i_2 \ldots \wedge X_t = i_t]$$

$$= \sum_{i_1 \ldots i_t \in B} \pi(i_1) W(i_1, i_2) W(i_2, i_3) \ldots W(i_{t-1}, i_t)$$

$$\overline{P} = \mathrm{Diag}(\mathbb{1}_B) \quad \begin{array}{c|cc} & B & \bar{B} \\ \hline B & I & 0 \\ \bar{B} & 0 & 0 \end{array}$$

$$(*) = \sum_{i_1 \ldots i_t} \pi(i_1) P(i_1, i_1) W(i_1, i_2) P(i_2, i_2) \ldots W(i_{t-1}, i_t) P(i_t, i_t)$$

$$= |\pi P (WP)^{t-1}|_1$$

$$= \langle P(WP)^{t-1} \mathbb{1}, \mathbb{1} \rangle_\pi$$

$$= \langle P(WP)^{t-1} \mathbb{1}, \mathbb{1}_B \rangle_\pi$$

$$\leq \|P(WP)^{t-1}\mathbb{1}\|_{2,\pi} \cdot \|\mathbb{1}_B\|_{2,\pi}$$

$$= \|(PWP)^{t-1}P\mathbb{1}\|_{2,\pi} \cdot \sqrt{\mu} \quad \text{Since } (P^2 = P)$$

$$= \|(PWP)^{t-1}\mathbb{1}_B\|_{2,\pi} \cdot \sqrt{\mu}.$$

For a general sq matrix M

$$\|M\|_{2,\pi} = \max_x \frac{\|Mx\|_{2,\pi}}{\|x\|_{2,\pi}}$$

= (largest eigen value)

<u>Obs:</u> $\|M_1 + M_2\|_{2,\pi} \leq \|M_1\|_{2,\pi} + \|M_2\|_{2,\pi}$

$$(\ast) \leq \|(PWP)^{t-1} \cdot \mathbb{1}_B\|_{2,\pi} \cdot \sqrt{\mu}$$

$$\leq \|(PWP)^{t-1}\|_{2,\pi} \|\mathbb{1}_B\|_{2,\pi} \cdot \sqrt{\mu}$$

$$= \|PWP\|_{2,\pi}^{t-1} \cdot \mu$$

To complete proof, suffices to show

$$\|PWP\|_{2,\pi} \leq \mu + \omega(1-\mu)$$
$$= (1-\omega)\mu + \omega$$

Warmup case: W - random walk
- independent according to π

$$W_{ind} = \begin{bmatrix} - & \pi & - \\ - & \pi & - \\ & \vdots & \\ - & \pi & - \end{bmatrix} = J\Pi$$

$$\Pi = \mathrm{Diag}(\pi)$$

$$P W_{ind} P = \begin{array}{c|c} & \begin{array}{cc} B & \bar{B} \end{array} \\ \begin{array}{c} B \\ \bar{B} \end{array} & \begin{bmatrix} \begin{matrix} -\pi/_B- \\ -\pi/_B- \\ -\pi/_B- \end{matrix} & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} \mathbb{1}_B \\ \\ \end{bmatrix} = \pi(B) \begin{bmatrix} \mathbb{1}_B \\ \\ \end{bmatrix}$$

Hence, $\|P W_{ind} P\|_{2\pi} = \pi(B) = \mu$

General Setting: W - random walk.

$\mathbb{1} = v_1, \ldots, v_n$ - e.vectors

$1, \omega_2, \ldots, \omega_n \geq -1$ e.values

$$W: \begin{cases} v_1 \longmapsto v_1 \\ v_2 \longmapsto \omega_2 v_2 \\ \vdots \\ v_n \longmapsto \omega_n v_n \end{cases}$$

$$W_{ind}: \begin{cases} v_1 \longmapsto v_1 \\ v_2 \longmapsto 0 \\ \vdots \\ v_n \longmapsto 0 \end{cases} \qquad W_E: \begin{cases} v_1 \longrightarrow \omega v_1 \\ v_2 \longmapsto \omega_2 v_2 \\ v_3 \longmapsto \omega_3 v_3 \\ v_n \longmapsto \omega_n v_n \end{cases}$$

$$W = (1-\omega) W_{ind} + W_E$$

Matrix Decomposition

Let $x = \sum \alpha_i v_i$; $\|x\|_{2,\pi}^2 = \sum \alpha_i^2$

$$W_E x = \alpha_1 \omega v_1 + \sum_{i=2}^{n} \alpha_i \omega_i v_i$$

$$\|W_E x\|_{2,\pi}^2 = \sum_{i=2}^{2} \alpha_i^2 \omega_i^2 + \alpha_1^2 \omega^2 \le \omega^2 \sum_{i=1}^{n} \alpha_i^2$$

$$\le \omega^2 \|x\|_{2,\pi}^2$$

Hence, $\|W_E\|_{2,\pi} \le \omega$

Return to proof

$$\|PWP\|_{2,\pi} = \|P((1-\omega)W_{ind} + W_E)P\|_{2,\pi}$$

$$\le (1-\omega)\|PW_{ind}P\|_{2,\pi} + \|PW_EP\|_{2,\pi}$$

$$\le (1-\omega)\cdot\mu + \|P\|_{2,\pi}\|W_E\|_{2,\pi}\cdot\|P\|_{2,\pi}$$

$$\le (1-\omega)\mu + \omega.$$

Hence, $\|PWP\|_{2,\pi} \le (1-\omega)\mu + \omega$

$$= \mu + \omega(1-\mu)$$

$\boxtimes$

Lovasz Theta Function.

1979: On the Shannon Capacity of a Graph.

G — $\alpha(G)$ — size of largest independent set

$\omega(G)$ — size of largest clique

$(\omega(\bar{G}) = \alpha(G))$

$\chi(G)$ – chromatic number of graph.

(Obs: $\omega(G) \le \chi(G)$)

$\bar{\chi}(G)$ – clique cover number

min # cliques that partition the vertex set

$(\bar{\chi}(G) = \chi(\bar{G}))$

Obs: $\alpha(G) \le \bar{\chi}(G)$

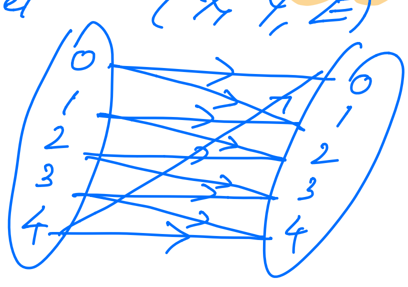
$\vartheta(G)$ – Lovasz Theta function

- $\vartheta(G)$ – efficiently computable (SDP formulation)
- Sandwich Theorem

$$\alpha(G) \le \vartheta(G) \le \chi(\bar{G})$$

- Shannon Capacity of a Graph.

Channel $(\mathcal{X}, \mathcal{Y}, E)$



$\mathcal{X}$ input

$\mathcal{Y}$ output

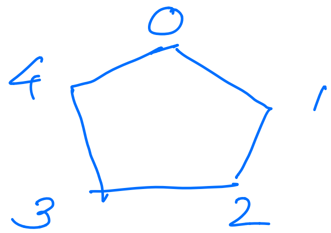
Confusability Graph

$V = \mathcal{X}$

$(x_1, x_2) \in E$ & $(x_1 \neq x_2)$

iff $\exists y \in \mathcal{Y}$

$(x_1 \to y)$ & $(x_2 \to y)$



C_5

For 1-letter words
$\alpha(G)$ – largest number of words I can transmit on the channel w/o error.

$\alpha(C_5) = 2$

Suppose 2-letter words.

00, 13, 24, 32, 43

G – confusability graph.

$G \boxtimes H$ – strong product

$G = (V, E_1)$; $H = (W, E_2)$

$V(G \boxtimes H) = V \times W$

$((u_1, u_2), (v_1, v_2)) \in E(G \boxtimes H)$

if (i) $(u_1, u_2) \neq (v_1, v_2)$

(ii) $\forall i \in [2]$, $u_i = v_i$ or $(u_i, v_i) \in E_i$

$$G^{\boxtimes k} = \begin{cases} G & \text{if } k = 1 \\ G^{\boxtimes (k-1)} \boxtimes G & \text{for } k > 1 \end{cases}$$

<u>Obs</u>: $G^{\boxtimes k}$ – confusability graph for k-letter words.

Max # of k-letter words that can be transmitted on this channel error free $= \alpha(G^{\boxtimes k})$

<u>Claim</u> $\alpha(G \boxtimes H) \geq \alpha(G) \cdot \alpha(H)$

Shannon Capacity of a graph G.

$$\sigma(G) = \sup_k \frac{\log \alpha(G^{\boxtimes k})}{k} = \lim_k \frac{\log(G^{\boxtimes k}}{k}$$

$\overline{C_5}$: $\quad \alpha(C_5) = 2$

$\alpha(C_5^{\boxtimes 2}) \geq 5 \quad \Rightarrow \quad \sigma(C_5) \geq \frac{\log 5}{2}.$

$$\alpha(G) \leq \sigma(G) \leq \log|V|$$

<u>Shannon:</u> What is $\sigma(C_5)$?

$$\Sigma(G) = \lim_{k \to \infty} \left(\alpha(G^{\boxtimes k})\right)^{1/k} = 2^{\sigma(G)}$$

Thm [Lovasz] $\Sigma(C_5) = \sqrt{5}.$

What do we know

$$\sqrt{5} \leq \Sigma(C_5) \leq 5$$

Suppose there exists a function f on graphs.

- (i) Upper Bd on Independence Number.

$$\alpha(G) \leq f(G)$$

- (ii) Sub-multiplicative

$$f(G \boxtimes H) \leq f(G) \cdot f(H)$$

then $\Sigma(G) \leq f(G)$

Pf:
$$\Sigma(G) = \sup_k \left(\alpha(G^{\boxtimes k})\right)^{1/k}$$
$$\leq \sup_k \left(f(G^{\boxtimes k})\right)^{1/k}$$
$$\leq \sup_k \left(\left(f(G)\right)^k\right)^{1/k} = f(G) \quad \boxtimes$$

Clique Cover number

$\bar{\chi}(G)$ = chromatic # of $\bar{G}$

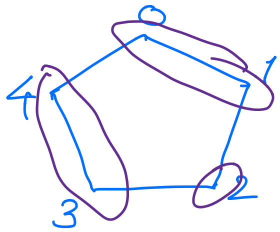
satisfies

(i) $\alpha(G) \leq \bar{\chi}(G)$

(ii) $\bar{\chi}(G)$ — submultiplicative (ie, $\bar{\chi}(G \boxtimes H) \leq \bar{\chi}(G) \cdot \bar{\chi}(H)$)

Hence, $\Sigma(G) \le \bar{\chi}(G)$.

$\overline{C_5}$: $\bar{\chi}(C_5) =$



$$\left.\begin{array}{l}\bar{\chi}(G) \le 3 \\ \bar{\chi}(G) > 2\end{array}\right\} \bar{\chi}(C_5) = 3$$

Hence, $\sqrt{5} \le \Sigma(C_5) \le 3$.

Next time:

Lovasz
- introduces orthonormal representations of a graph
- define the $\vartheta(G)$.
- $\vartheta(G) \ge \alpha(G)$
- $\vartheta(G \boxtimes H) \le \vartheta(G) \cdot \vartheta(H)$
- $\vartheta(C_5) \le \sqrt{5}$.
- SDP formulation for $\vartheta(G)$
- Sandwich theorem

  $\alpha(G) \le \vartheta(G) \le \bar{\chi}(G)$