Today
- Random Walk
- Hitting Set Lemma.

CS5.413.1
Pseudorandomness
Lecture 09 (2021-9-21)
Instructor: Prahladh Harsha.

Recap from last time.

① A - adjacency matrix

$M = D^{-1}A \qquad D = \text{Diag}(\deg)$

$M(u,v) = \Pr[u \mapsto v]$) Random Walk matrix

Right Multiplication: p - prob dist on V

$p^T \to p^T M \to p^T M^2 \to \ldots$

π - stationary dist. $\qquad \pi^T M = \pi$

$$\pi(u) = \frac{\deg(u)}{\sum_{w \in V} \deg(w)}$$

Left Multiplication: $f \mapsto Mf$

Averaging Operator

$M \mathbb{1} = \mathbb{1}$.

2) Inner Product on $\mathbb{R}^V$

$$\langle f,g\rangle_\pi = \mathbb{E}_{u\sim\pi}\left[f(u)g(u)\right]$$

$$= \sum_{u\in V}\pi(u)f(u)g(u) = f^T\Pi g$$

where $\Pi = \mathrm{Diag}(\pi)$

M- self adjoint w.r.t $\langle\, ,\, \rangle_\pi$

$$\langle f, Mg\rangle_\pi = \langle Mf, g\rangle_\pi \quad , \forall f,g \in \mathbb{R}^V$$

(In matrix notation,

$$f^T\Pi M g = f^T M^T \Pi g \quad \forall f,g\in\mathbb{R}^V$$

ie, $\Pi M = M^T\Pi$

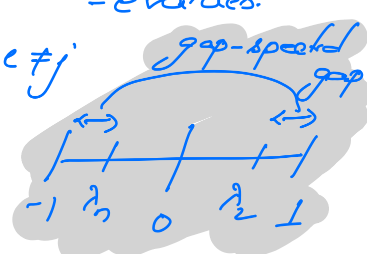
Eigen-basis:

$\mathbb{1} = v_1 \quad , v_2 \ldots \qquad v_n$ - evectors $\in \mathbb{R}^V$

$1 \quad , \lambda_2, \ldots \qquad , \lambda_n \geq -1$ - evalues.

Spectral gap $= \min\{1-\lambda_2, \lambda_n-(-1)\}$

s.t $\begin{cases} \langle v_i, v_j\rangle_\pi = 0 & , i\neq j \\ \langle v_i, v_i\rangle_\pi = 1 \\ Mv_i = \lambda_i v_i. \end{cases}$



3) Expander Mixing Lemma:

M- r.w matrix w/ spectral gap $1-\lambda$.

& stationary dist π

$S, T \subseteq V$, $\pi(S) = \alpha$; $\pi(T) = \beta$

$$\left| \Pr_{e=(u,v)\sim E} [u \in S, v \in T] - \alpha\beta \right| \leq \lambda \sqrt{\alpha(1-\alpha)\beta(1-\beta)}$$

—

Applying to S , $T = V \setminus N(S)$.
we get

(4) Lemma: [Spectral Expansion $\Rightarrow$ Vertex Expansion]
$G = (V, E)$ w/ spectral gap $1-\lambda$,

$\Downarrow$

G is $\left(\rho N, \frac{1}{\lambda^2 + (1-\rho)\lambda^2}\right)$ - vertex expander. $\forall \rho \in (0,1)$

Cor: G is a D-regular on N-vertices w/ spectral gap $\gamma < 1$

$\Downarrow$

G is $\left(\frac{N}{2}, 1+\delta\right)$ - vertex expander for some $\delta > 0$.

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Surprisingly, the converse is also true.

Lemma (Vertex Expansion $\Rightarrow$ Spectral Expansion)

For every $\delta > 0$ & $D > 0$, there exists $\lambda > 0$ s.t

If G is a D-regular $\left(\frac{N}{2}, 1+\delta\right)$-vertex expander then G is $(1-\lambda)$-spectral expander, $\left(\lambda = \Omega\left(\frac{\delta}{D}\right)^2\right)$.

Thm: Let $\mathcal{G}$ be a infinite family of D-regular graphs, then the following two are equivalent.

(1) $\exists\ \delta > 0$, $G \in \mathcal{G}$ is $\left(\frac{N}{2}, 1+\delta\right)$-vertex expander

(2) $\exists\ 0 < \lambda < 1$, $G \in \mathcal{G}$ is a $(1-\lambda)$-spectral expander

Spectral $\rightarrow$ Vertex

G is $(1-\lambda)$-spectral expander

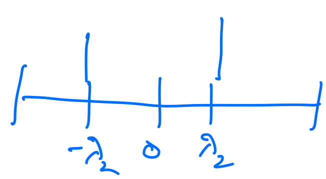
$$\Downarrow$$

G is $\left(\rho N, \frac{1}{\lambda^2 + \rho(1-\lambda^2)}\right)$- vertex expander.

If G is D-regular, $\frac{1}{\lambda^2+\rho(1-\lambda^2)} < D$

$\rho \rightarrow 0$; $\lambda^2 > \frac{1}{D}$

(i.e., $\lambda > \frac{1}{\sqrt{D}}$)



Slight Improvement (16 for λ)

Thm [Alon-Boppana].

Let G be an infinite family of D-regular w/ spectral expansion $1-\lambda$, then $\lambda(G_N) \geq \frac{2\sqrt{D-1}}{D} - O_N(1)$

where $O_N(1) \to 0$ as $N \to \infty$.

Thm [Friedman]

For any constant $D \geq 3$, a random D-regular graph on N vertices then G is a $(1-\lambda)$-spectral expander where $\lambda \leq \frac{2\sqrt{D-1}}{D} + O_N(1)$

with high probability $(1-O_N(1))$

(where $O_N(1) \to 0$ as $N \to \infty$)

Lubotsky-Philips-Sarnak

- gave an explicit construction of D-regular expanders satisfying

$\lambda(G) \leq \frac{2\sqrt{D-1}}{D}$ } Ramanujan Graphs.

Random Walks

M - random walk matrix

p_0 - initial distribution on vertices

$p_1 = M^T p_0$

$\vdots$

$p_{t+1} = M^T p_t$

Qn: Does p_t converge to π as $t \to \infty$.

YES. If the matrix M has spectral gap $\geq \lambda$.

Total variation distance between p_t & π.

$$d_{TV}(p_t, \pi) = \frac{1}{2} \sum_{v \in V} |p_t(v) - \pi(v)|$$

$$= \frac{1}{2} \sum_{v \in V} \pi(v) \left| \frac{p_t(v)}{\pi(v)} - 1 \right|$$

$$= \frac{1}{2} \sum_{v \in V} \pi(v) \left| (\Pi^{-1} p_t)(v) - 1 \right|$$

$$= \frac{1}{2} \left\| \pi^{-1} p_t - \mathbb{1} \right\|_{1,\pi}$$

$$\|f\|_{k,\pi} = \left(\sum \pi(v) |f(v)|^k \right)^{1/k}$$

$$\leq \frac{1}{2} \left\| \pi^{-1} p_t - \mathbb{1} \right\|_{2,\pi}$$ (Norms increases as k increases)

Study: $\left\| \pi^{-1} p_t - \mathbb{1} \right\|_{2,\pi}$ vs $\left\| \pi^{-1} p_{t+1} - \mathbb{1} \right\|_{2,\pi}$

where $p_{t+1} = M^T p_t$

Lets write.

$$\pi^{-1} p_t = \alpha \mathbb{1} + v$$ where $v \perp_\pi \mathbb{1}$

$$\langle \mathbb{1}, \pi^{-1} p_t \rangle_\pi = \alpha \langle \mathbb{1}, \mathbb{1} \rangle_\pi$$

$$\sum \pi(v)\, 1\, \frac{1}{\pi(v)} p_t(v) = \alpha$$

$\alpha = 1$ if p_t is a prob distribution

$$\pi^{-1} p_t = \mathbb{1} + v$$ where $v \perp_\pi \mathbb{1}$.

$\Pi^{-1} p_t - \mathbb{1} = v$ and $v \perp_\pi \mathbb{1}$

Let's understand what happens we take a random step.

$$\Pi^{-1} p_{t+1} - \mathbb{1} = \Pi^{-1} M^T p_t - \mathbb{1}$$

$$= M \Pi^{-1} p_t - \mathbb{1} \quad (\text{Since } \Pi M = M^T \Pi)$$

$$= M(\Pi^{-1} p_t - \mathbb{1}).$$

$$= Mv$$

$$\|\Pi^{-1} p_t - \mathbb{1}\|_{2,\pi} \quad \text{vs} \quad \|\Pi^{-1} p_{t+1} - \mathbb{1}\|_{2,\pi}$$

is

$$\|v\|_{2,\pi} \quad \text{vs} \quad \|Mv\|_{2,\pi}$$

for some $v \perp_\pi \mathbb{1}$.

Because M has spectral gap $1-\lambda$

for all $v \perp_\pi \mathbb{1}$

$$\|Mv\|_{2,\pi} \leq \lambda \|v\|_{2,\pi}.$$

Putting them together.

$$d_{TV}(P_t, \pi) \le \frac{1}{2} \| \pi^{-1} P_t - \mathbb{1} \|_{2,\pi}$$

$$\le \frac{\lambda^t}{2} \| \pi^{-1} P_0 - \mathbb{1} \|_{2,\pi} . \quad \ldots (*)$$

If $\lambda < 1$, then $d_{TV}(P_t, \pi) \to 0$ as $t \to \infty$.

Let's try to understand $(*)$ when G is D-regular. π - uniform dist

$$\| \pi^{-1} P_0 - \mathbb{1} \|_{2,\pi}^2 = \sum \frac{1}{N} \left(P_0(v) N - 1 \right)^2$$

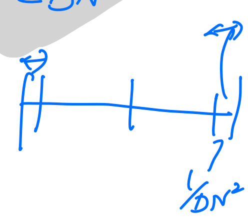
Assume P_0 - start dist is conc on a vertex

ie $P_0 = (0, . \; 0, 1, 0, \ldots 0)$

$$\| \pi^{-1} P_0 - \mathbb{1} \|_{2,\pi}^2 = \sum \frac{1}{N} (N-1)^2 = O(N)$$

<u>Lemma</u>: Any D-regular graph on N vertices that is connected & non-bipartite satisfies $\gamma \ge \frac{1}{CDN^2}$.

(ie, $\lambda \le 1 - \frac{1}{CDN^2}$)



$\lambda \leq \left(1 - \frac{1}{CN^2 D}\right)$ $\qquad$ $t = O(CDN^2 \log N)$.

$$d_{TV}(P_t, \pi) \leq \frac{\lambda^t}{2} \cdot N = O\left(\frac{1}{N^{100}}\right)$$

Hence, in $O(CDN^2 \log N)$ steps the random walk on any D-regular graph converges to the uniform dist.
(if the graph is connected & non-bipartite)
in polynomial # steps.

What about random walk on expanders?

Any D-regular graph (connected & non-bipartite) $\Rightarrow \gamma \geq \frac{1}{CDN^2}$

(ie, $\lambda \leq 1 - \frac{1}{DN^2}$)

Expander $\Rightarrow$ $\gamma \geq c > 0$

$\underbrace{\lambda \leq 1 - c < 1}_{\text{constant.}}$

As λ is a constant bdd away from 1, $O(\log N)$ steps suffice to converge to the uniform dist^n

\# random coins in order to take an $t = O(\log N)$-step walk starting from a fixed point.

$= t \cdot \log D$

$= \log N \cdot \log D$

$= O(\log N)$ if D is constant.

Today:
Random walks on expander converge to the uniform dist^n in $O(\log N)$ steps even if we start from a fixed vertex.

Next time:
t - Random walk starting at a uniformly random vertex has the property.

"t steps look "simultaneously random""