Pseudorandomness - Lecture 24.

Instructor: Ramprasad
Date: 2021-11-23
Lecture #: 24.

Agenda: Pseudorandomness in other contexts

- ▷ Szemeredi's Regularity lemma
- ▷ Szemeredi's thm on k-term APs in dense sets.
- ▷ Towards the Green-Tao theorem (APs within primes)

AP: $x, x+d, x+2d, \ldots, x+(k-1)d$.

Qn: Fix any k and r. Suppose you colour $\mathbb{N}$ using r colours, will you always find a monochromatic k-AP?

[van der Warden] : Yes.

Erdős: Is it just because one of the colour classes has density $\geq 1/r$?

Conj: For all δ, k, as long as N is large enough, any subset $A \subseteq [N]$ with $|A| = \delta N$ contains a k-AP.

Roth's Theorem (1953): Above conj for $k=3$. (Fourier analysis)

Szemeredi's Thm (1975): Above conj for all k.

Green-Tao Thm (2004): Even within primes, there are long APs.

"structure + randomness"

Szemeredi's regularity lemma: "Any graph can be broken up into a constant number of parts s.t most pairs behave pseudorandomly".

EML: For all $A, B \subseteq V$

$$\left| \frac{E(A,B)}{|E|} - \mu(A)\mu(B) \right| \leq \lambda \cdot \sqrt{\mu(A)\mu(B)}$$

$$d(S,T) = \frac{E(S,T)}{|S| \cdot |T|} \quad \text{edge density.}$$

Defn: (ε-regularity for a pair). (A,B) is said to be an ε-reg pair of subsets if $\forall\, U \subseteq A$, $W \subseteq B$ with $|U| \geq \varepsilon |A|$ and $|W| \geq \varepsilon |B|$, we have

$$|d(A,B) - d(U,W)| \leq \varepsilon.$$

Defn: (ε-regular partition) A partition $\mathcal{P}$: $V(G) = V_1 \sqcup \dots \sqcup V_k$ is an ε-regular partition if

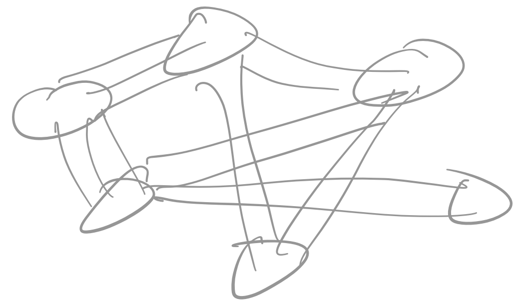
$$\sum_{\substack{(i,j): \\ (V_i, V_j) \text{ is not} \\ \varepsilon\text{-regular}}} |V_i||V_j| \leq \varepsilon \cdot |V(G)|^2.$$

If $|V_i| \approx n/k$, then $\leq \varepsilon k^2$ pairs are not regular.

Thm: (Szemeredi's Regularity Lemma) For any $\varepsilon > 0$, there is a constant M (dep. on ε) such that any n-vertex graph has an ε-reg. partition into $\leq M$ parts.

Imp: M <u>does not</u> dep on n.

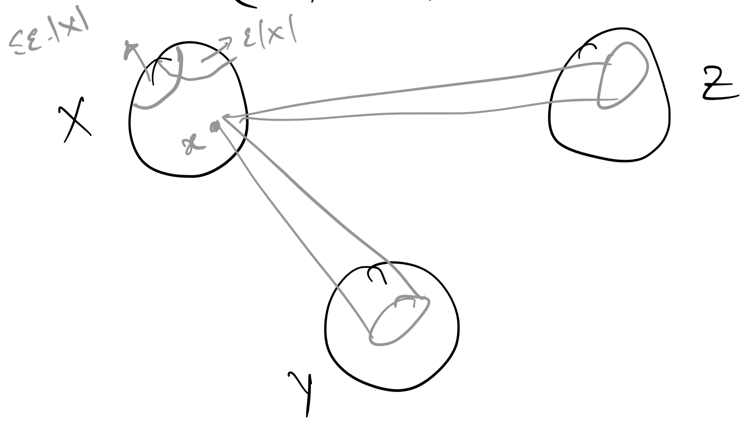
Very useful for any <u>dense</u> graph.



$M \approx 2^{2^{2^{2}}} \underset{\downarrow}{\uparrow} 1/\varepsilon^5$

Applications:

- If $X \sqcup Y \sqcup Z = V$ and all pairs are ε-regular, with $d_{XY}, d_{YZ}, d_{XZ} \geq 2\varepsilon$. Then, the # XYZ-triangles $\geq (1-2\varepsilon)(d_{XY}-\varepsilon)(d_{YZ}-\varepsilon)(d_{ZX}-\varepsilon)\,|X|.|Y|.|Z|$.



$\frac{1}{\delta} \approx 2^{2^{2^{2}}} \underset{\downarrow}{\uparrow} f(\varepsilon)$

Thm: (Rusza-Szemeredi) [Triangle Removal Lemma]. $\forall \varepsilon > 0\ \exists \delta > 0$. If G has $\leq \delta . n^3$ triangles, then there is some $\leq \varepsilon n^2$ edges removing which makes G triangle-free.

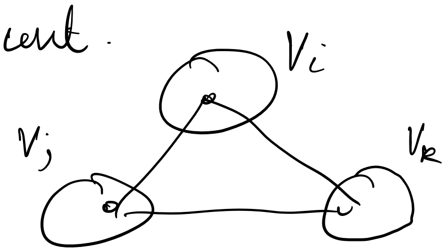
(If $\#\Delta s = o(n^3)$ then we can make it Δ-free by removing $o(n^2)$ edges).

Sketch: Step 1: Use SRL $G = V_1 \sqcup \dots \sqcup V_k$.

Step 2: Clean up:

- Throw away edges in irreg pairs.
- Throw away all "sparse" edges.
- Throw away all "small" parts.

Step 3: Count.



One $\Delta \Rightarrow$ many triangles!

$\square$.

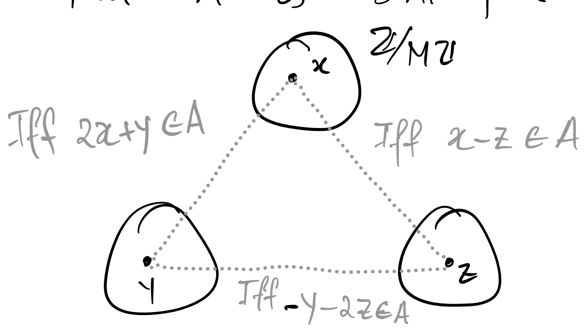
Cor: If you have a graph G s.t each edge is in exactly 1 triangle, then $\#\text{edges} = o(n^2)$.

Pf: $\#\Delta s = O(n^2) = o(n^3)$. Then G can be made Δ-free by removing $o(n^2)$ edges. But req $\geq |E|/3$ edges to rem.. $\square$.

Why do we care about these?

Roth's Thm: For all $\delta > 0$, and all N large enough, $A \subseteq [N]$ with $|A| \geq \delta . N \Rightarrow A$ contains a 3-AP.

Pf using TRL: Say A is a 3-AP free subset of $[N]$.
$M = 2N+1$ and think of A mod M.
Then A is 3-AP free even mod M.



(x,y,z) is a Δ

$\Updownarrow$

$2x+y,\ x-z,\ -y-2z \in A$

AP with c.d $-x-y-z$

If A was 3-AP free, $\Rightarrow$ only trivial APs

$\Rightarrow$ $x+y+z = 0 \bmod M$..

$\Rightarrow$ every edge is in exactly one triangle.

$\therefore$ $|E| = o(M^2) = o(N^2) = O(|A| \cdot M)$

$\Rightarrow$ $|A| = o(N)$

$\square$.

$\therefore$ Any set of constant density in $[N]$ has a 3-AP in it.

For $k=4$ (and higher):

- Work with a 4-partite 3-uniform with

$3w+2x+y$, $2w+x-z$, $w-y-2z$, $-x-2y-3z$.

- Work with the hypergraph removal lemma.

(req. a hypergraph version of SRL).

(If you have "few" copies of a hypergraph, then the can all be removed by removing "few" hyperedges).

Extensions and reformulations

- Counting version [Varnavides]

"Any const. density set must have many APs."

$\forall \delta > 0$ $\exists c > 0$ s.t any $A \subseteq [N]$ with $|A| \geq \delta N$ has $\geq c \cdot N^2$ 3-APs.

- Weighted version: $\forall N > N_0(\varepsilon, c)$

$\forall \delta > 0$ $\exists c > 0$ so that any $f: \mathbb{Z}_N \to [0,1]$ with $\mathbb{E}[f] \geq \delta$ satisfies. $\mathop{\mathbb{E}}_{x,d}\left[f(x) \cdot f(x+d) \cdot f(x+2d)\right] \geq c$.

Qn: How small can we make δ? How large are AP free subsets of $[N]$?

[Behrend] $|A| \geq N \cdot \exp(-\sqrt{\log N})$ that is 3-AP free.

$N \cdot \exp(-c\sqrt{\log N}) \leq$ size of largest 3AP-free subset of $[N]$ $\leq c \cdot \frac{(\log\log N)^4}{\log N} \cdot N$.

Are there sparse versions of these regularity statements?

TRL: $\forall \varepsilon, \exists \delta$ If G has $\leq \delta n^3$ triangles, then they can be removed using $\leq \varepsilon n^2$ edges.

Dream sparse version: $p = p(n)$.

$\forall \varepsilon, \exists \delta$: If G has $\leq \delta n^3 p^3$ triangles, then they can all be removed using $\leq \varepsilon n^2 p$ edges.

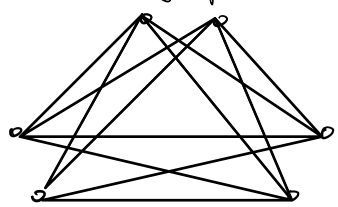
... just false! But it is true if G has const dens. in a "pseudorandom host graph"

Sparse Triangle Removal Lemma: [Conlon-Fox-Zhao] $p = p(n)$
$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t if Γ is a "sufficiently pseudorandom" host graph with edge density p, and G is a subgraph of Γ, then if G has $\leq \delta n^3 p^3$ triangles, then they can all be removed using $\leq \varepsilon \cdot n^2 p$ edges.

What is "sufficiently pseudorandom"?

- Γ contains the "right" density of embeddings of $K_{2,2,2}$ and its subgraphs

"2-blow up of a Δ"



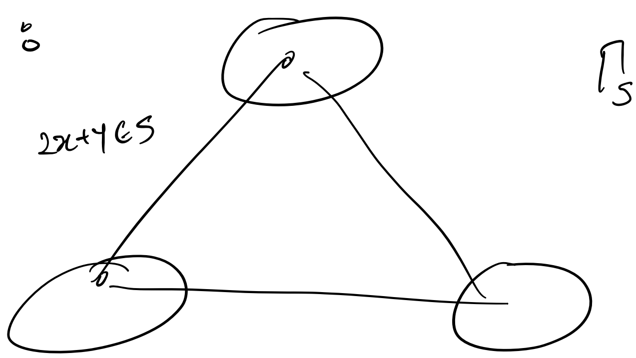
$\approx p^{12}$

Relative Roth's Theorem: [Conlon-Fox-Zhao]

Suppose $S \subseteq \mathbb{Z}/N\mathbb{Z}$ with $|S| = pN$ and S suff. p.r.
Then, any subset $A \subseteq S$ with $|A| \geq \delta . |S| = \delta . N . p$ contains $\geq c(\delta) . N^2 . p^3$ 3-term APs.

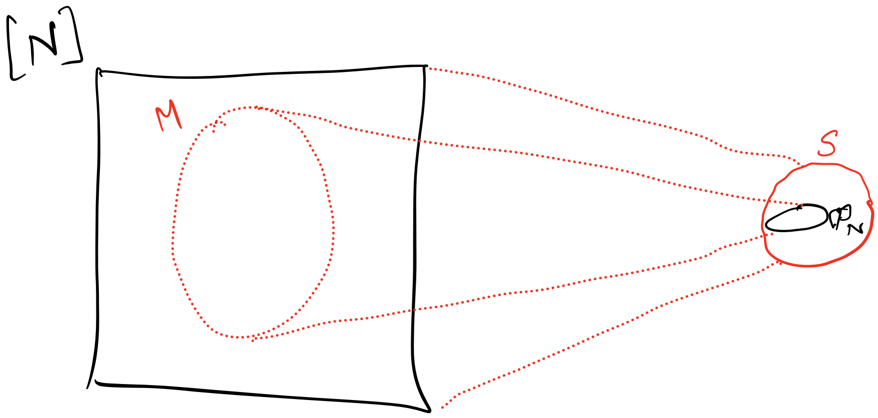
"If A is a const. density subset within a pseudorandom host S, then A has many 3-term APs".

S suff. PR:



[Green-Tao] $\mathcal{P}$: Primes $\quad \mathcal{P}_N = \mathcal{P} \cap [N] \quad \frac{|\mathcal{P}_N|}{N} \approx \frac{1}{\ln N}$

But if $\mathcal{P}_N$ has const. density in a pseudorandom set, then Rel. Roth's thm will give us the result.



$\delta = \frac{|\mathcal{P}_N|}{|S|} \approx \frac{|M|}{N}$

and # APs in $\mathcal{P}_N$ $\tilde{\approx}$ # APs inside M (upto scaling)

Green-Tao-Zeigler: The Dense Model Thm.

Next class: - The Dense Model Theorem
(a pf by Reingold-Trevisan-Tulsiani-Vadhan].)