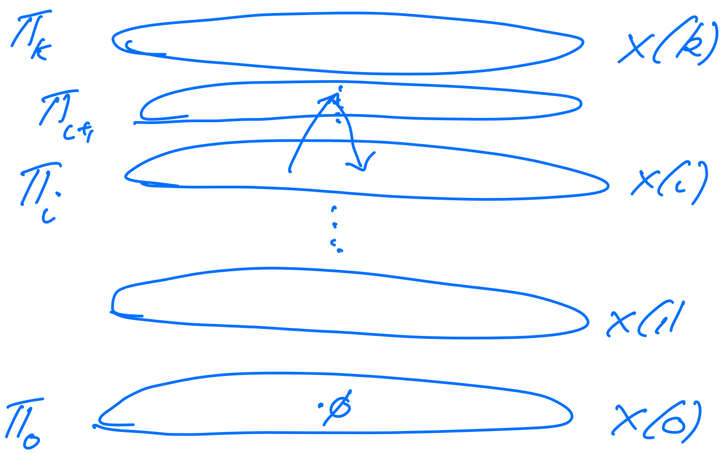
Today
- Sampling Spanning Trees (applⁿ of HDXs)

CS5.413.1 Pseudorandomness
Lecture 28 (2021-12-9)
Instructor: Prahladh Harsha.

Recall defⁿ of HDX

X — simplicial complex
$(X(0), X(1), \ldots, X(k))$
$X(0) = \{\emptyset\}$, $X(i)$ — sets of size i.
down-closed.



Up-down walk
- P_i^{Δ}
- $P_i^{\wedge}$ (non-lazy)

$$P_i^{\Delta} = \frac{1}{i+1} I + \frac{i}{i+1} P_i^{\wedge}$$

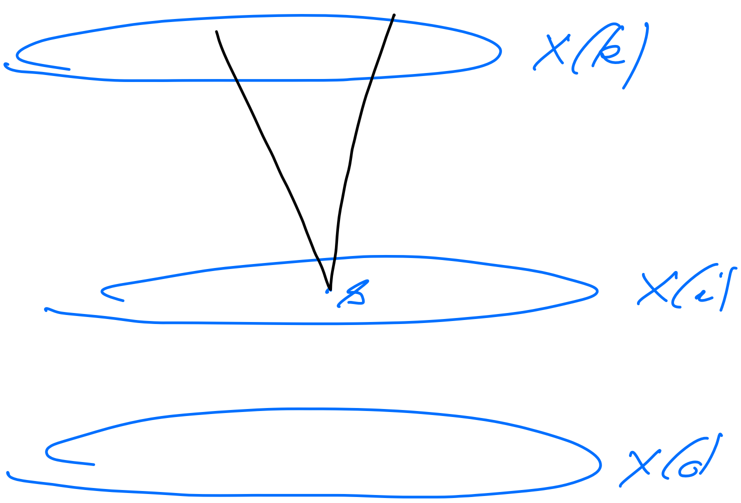
Down-Up walk:
- P_i^{∇} (lazy version)

$- P_i^\vee$ (non-lazy).

Defn: λ-HDX if $P_i^\wedge \approx_\lambda P_i^\triangledown$
(non-lazy up-down) down-up.

(ie $\| P_i^\wedge - P_i^\triangledown \| \leq \lambda$).

Alternate defn in terms of links



$s \in X(i)$

$$X_s = \{ t \setminus s \mid t \supseteq s;\ t \in X \}$$

$$X_s = \Big(\underset{\{\phi\}}{\underset{\shortparallel}{X_s(0)}},\ X_s(1) \dots ,\ X_s(k-i) \Big)$$

Defn: X is λ-link HDX if $\forall\, 0 \leq i \leq k-2$, $s \in X(i)$, the underlying graph of X_s is λ-expander.

Thm: X is λ-link HDX $\Rightarrow$ X is λ-HDX

Lemma: $P_k^{\wedge} - P_k^{\nabla} \preceq \lambda I$ if X is λ-link HDX.

(where $A \preceq B$ ie, $B - A \succeq 0$.
or equivalently
$\forall f \quad \langle f, Af \rangle \leq \langle f, Bf \rangle$

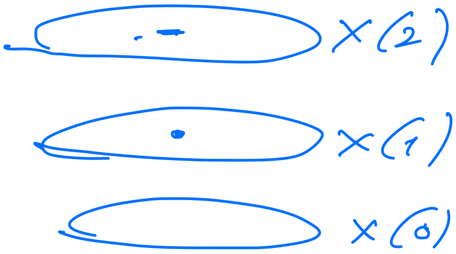
$\mathcal{F}(i) = \{ f : X(i) \to \mathbb{R} \}$.
equip $\mathcal{F}(i)$ w/ an inner product

$$\langle f, g \rangle_{\pi_i} = \mathbb{E}_{s \leftarrow \pi_i} [f(s) g(s)]$$ where
$f, g : X(i) \to \mathbb{R}$

$f, g : X(1) \to \mathbb{R}$.

$$\langle f, g \rangle = \langle f, g \rangle_{\pi_1}$$
$$= \mathbb{E}_{v \leftarrow \pi_1} [f(v) g(v)]$$
$$= \mathbb{E}_{\{u,v\} \leftarrow \pi_2} [f(v) g(v)]$$

$$= \mathbb{E}_{u \leftarrow \Pi_1} \ \mathbb{E}_{v \leftarrow \Pi_1^u} \left[f(v) g(v) \right]$$



$$= \mathbb{E}_{u \leftarrow \Pi_1} \langle f_u, g_u \rangle$$

where $f_u = f\big|_{X_u(1)}$

Hence $\langle f, g \rangle = \mathbb{E}_{u \leftarrow \Pi_1} \langle f_u, g_u \rangle$

X - complex

A - normalized adjacency matrix of the underlying graph $(X(0), X(1), X(2))$
Π_2.

$f, g : X(1) \to \mathbb{R}$.

$$\langle Af, g \rangle = \mathbb{E}_{\{u,v\} \leftarrow \Pi_2} \left[f(u) g(v) \right]$$

$$= \mathbb{E}_{\{u,v,w\} \leftarrow \Pi_3} \left[f(u) g(v) \right]$$

$$= \mathbb{E}_{w \leftarrow \Pi_1} \ \mathbb{E}_{\{u,v\} \leftarrow \Pi_2^w} \left[f(u) g(v) \right].$$

$$= \mathop{\mathbb{E}}_{\omega \leftarrow \pi_1} \langle A_\omega f_\omega, g_\omega \rangle$$

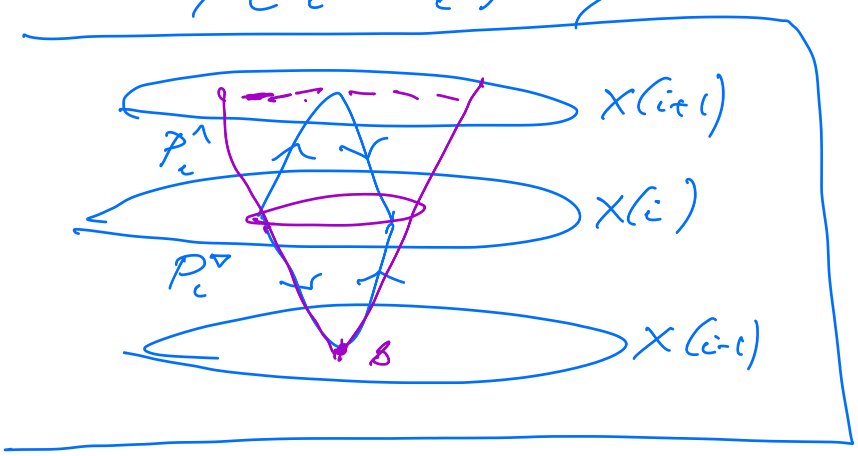
where A_ω is the (norm) adj matrix of the underlying graph of the link X_ω.

$$\langle Af, g \rangle = \mathop{\mathbb{E}}_{\omega \leftarrow \pi_1} \langle A_\omega f_\omega, g_\omega \rangle$$

Lemma: $P_i^\wedge - P_i^\nabla \preceq \lambda I$ if X is λ-link HDX.

Proof: $f : X(i) \to \mathbb{R}$

$$\langle f, (P_i^\wedge - P_i^\nabla) f \rangle$$



$$\langle f, P_i^\nabla f \rangle$$

$$= \mathop{\mathbb{E}}_{t \leftarrow X(i)} \left[f(t) (P_i^\nabla f)(t) \right]$$

$$= \mathop{\mathbb{E}}_{s \leftarrow X(i-1)} \mathop{\mathbb{E}}_{\substack{t, t' \leftarrow X(i) \\ t, t' \supseteq s}} \left[f(t) f(t') \right]$$

$$\langle f, \underbrace{(P_c^{\wedge} - P_c^{\triangledown})}\, f\rangle = \mathbb{E}_{s \leftarrow X(i-1)} \left[\langle f_s, (A_s - J_s) f_s \rangle \right]$$

where A_s - (norm) adj matrix of the underlying graph of the link X_s

$$(P_c^{\triangledown} f)(t) = \mathbb{E}_{\substack{s \subseteq t \\ t' \supseteq s}} [f(t')]$$

$$\langle f, P_c^{\triangledown} f\rangle = \mathbb{E}_t \Big[f(t) \mathbb{E}_{\substack{s \subseteq t \\ t' \supseteq s}} f(t')\Big]$$

$$= \mathbb{E}_s \mathbb{E}_{t, t' \supseteq s} [f(t) f(t')]$$

$$\langle f, P_c^{\wedge} f\rangle = \mathbb{E}_s \mathbb{E}_{\{t, t'\} \leftarrow X_s(2)} [f(t) f(t')]$$

$$\langle f, (P_c^{\wedge} - P_c^{\triangledown}) f\rangle = \mathbb{E}_{s \leftarrow X(i-1)} \langle f_s, (A_s - J_s) f_s\rangle$$

Recall λ is an upper bd on

2^{nd} e-value of A_S.

Hence for all $g_S \perp \mathbb{1}_S$; $\langle g_S, A_S g_S \rangle \leq \lambda \langle g_S, g_S \rangle$

$f_S = \alpha \mathbb{1}_S + f_S^\perp \qquad \alpha \mathbb{1}_S = J_S f_S$

$$\langle f_S, (A_S - J_S) f_S \rangle = \langle \alpha \mathbb{1}_S + f_S^\perp, (A_S - J_S) f_S \rangle$$

$$= \langle \alpha \mathbb{1}_S + f_S^\perp, A_S f_S^\perp \rangle$$

$$= \langle f_S^\perp, A_S f_S^\perp \rangle$$

$$\leq \lambda \langle f_S^\perp, f_S^\perp \rangle$$

$$\leq \lambda \langle f_S, f_S \rangle$$

2^{nd} e-value bound appears.

$(A_S - J_S)(\alpha \mathbb{1}_S + f_S^\perp)$

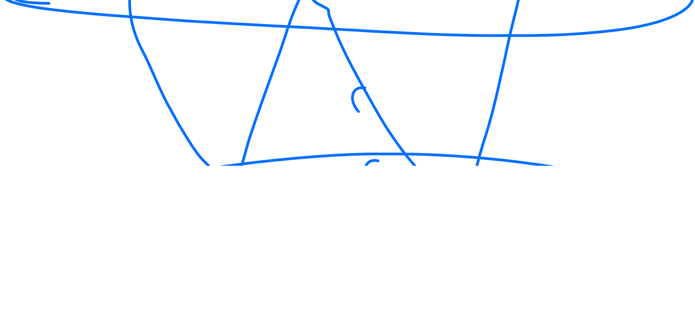
$= A_S f_S^\perp + \alpha A_S \mathbb{1}_S - \alpha \mathbb{1}_S + 0$

$= A_S f_S^\perp$

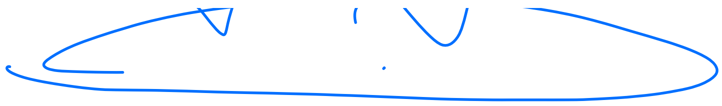
Hence $\quad (P_c^\wedge - P_c^\triangledown) \preceq \lambda I \quad \boxtimes$

<u>Return</u>: to Sampling of spanning trees

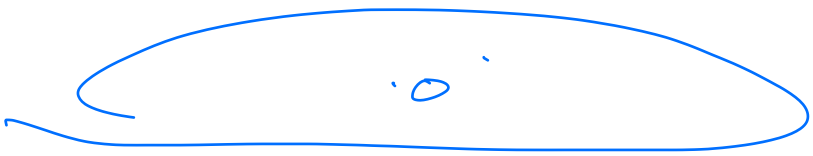
$\mathcal{F}$ = set of all forest



$\mathcal{F}(n-1)$ - spanning



$\mathcal{F}(n-2)$



$\mathcal{F}(0)$

Glauber Dynamics Walk:

Down-Up walk of $\mathcal{F}(n-1)$

How well P^{∇}_{n-1} mixes?

In particular, we want to understand $\lambda_2(P^{\nabla}_{n-1})$

Lemma 1: $P^{\wedge}_i - P^{\nabla}_i \preceq \lambda I$

~~Cor~~ **Cor:** $\lambda_2(P^{\wedge}_i) \le \lambda + \lambda_2(P^{\nabla}_i)$

What we just proved.

Lemma 2: $\mathcal{F}$ is 0-link HDX

(i.e., for all $s \in \mathcal{F}(i)$, $0 \le i \le n-2$.

2nd e-value of the underlying graph of the link $\mathcal{F}_s \leq 0$.

$$1 - \lambda_2\left(P^{\nabla}_{n-1}\right)$$

$$= 1 - \lambda_2\left(P^{\Delta}_{n-2}\right)$$

$$= 1 - \lambda_2\left(\frac{1}{n-1} I + \frac{n-2}{n-1} P^{\wedge}_{n-2}\right)$$

$$= \frac{n-2}{n-1} - \frac{n-2}{n-1} \lambda_2\left(P^{\wedge}_{n-2}\right)$$

$$= \left(\frac{n-2}{n-1}\right)\left(1 - \lambda_2\left(P^{\wedge}_{n-2}\right)\right)$$

(Assume:

$$\lambda_2\left(P^{\nabla}_{i}\right) = \lambda_2\left(P^{\Delta}_{i-1}\right)$$

$$\geq \frac{n-2}{n-1}\left(1 - \lambda_2\left(P^{\nabla}_{n-2}\right)\right)$$

$$\geq \frac{n-2}{n-1} \cdot \frac{n-3}{n-2}\left(1 - \lambda_2\left(P^{\nabla}_{n-3}\right)\right)$$

$\vdots$

$$\geq \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2}\left(1 - \lambda_2\left(P^{\nabla}_{1}\right)\right)$$

$$= \frac{1}{n-1}\left(1 - \lambda_2\left(P^{\nabla}_{1}\right)\right) = \frac{1}{n-1}$$

Spectral gap of P_{n-1}^{∇} is at least $\frac{1}{n-1}$

Markov Mixing:

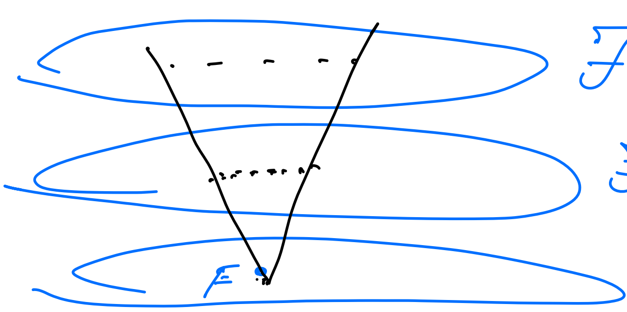
If a random walk has spectral gap at least γ.
then the walk mixes in time proportional to $O\left(\frac{1}{\gamma}\right)$

(Formally $t_{mix}(\varepsilon) \leq \frac{1}{\gamma} \log\left(\frac{1}{\varepsilon \cdot \pi_{min}}\right)$.)

Proof of Lemma 2:

ie $\mathcal{F}$ is 0-link HDX.

(Key Observation of Anari-Charan-Liu-Vinzant.



$\mathcal{F}(n-1)$ – spanning
$\mathcal{F}(n-2)$ – forests of size $n-2$
$\mathcal{F}(n-3)$

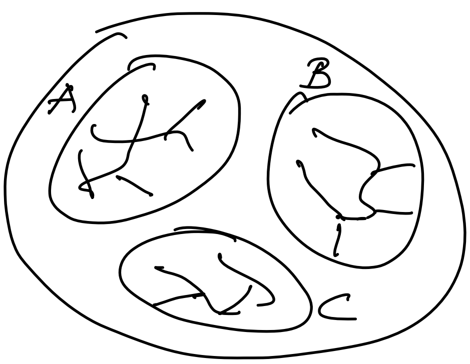
↳ forests of size $n-3$

Let $F \in \mathcal{F}(n-3)$ be a forest of size $n-3$.

Look at link $\mathcal{F}_F$

& see how the underlying graph of $\mathcal{F}_F$ looks like.

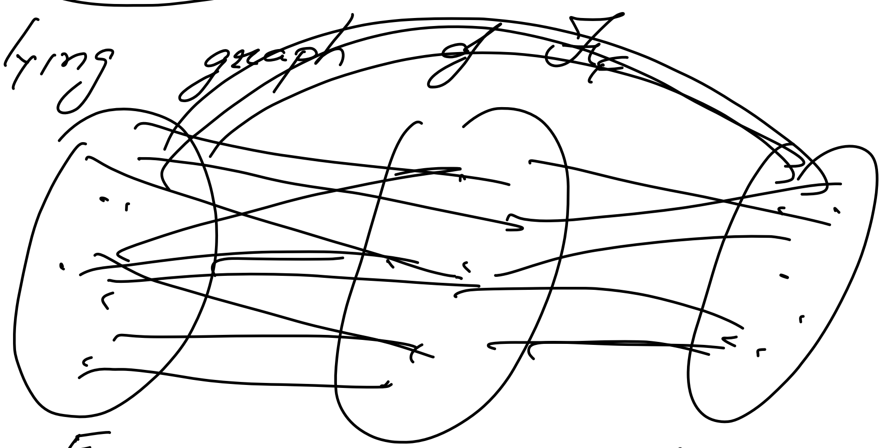
G



$G \setminus F$

$\mathcal{F}_F(1)$ - vertices of the link $\mathcal{F}_F$

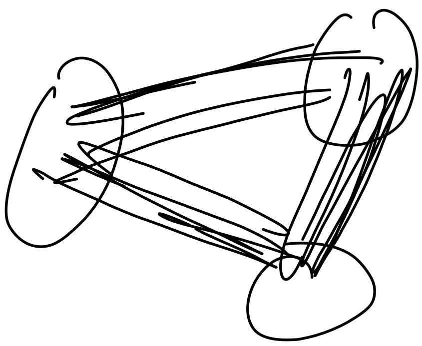
Underlying graph of $\mathcal{F}_F$



$F_{AB} = E_G(A,B)$ $\quad$ $F_{BC} \cong E_G(B,C)$ $\quad$ $F_{CA} \cong E_G(C,A)$

Underlying graph of $\mathcal{F}_F$ is the complete 3-partite graph

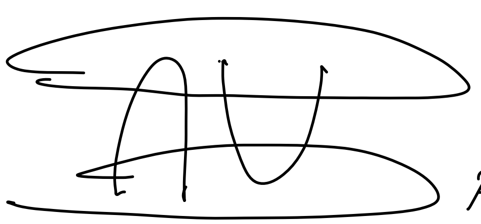
$E_G(A,B)$ $\quad E_G(B,C)$ $\quad E_G(C,A)$



Complete k-partite graph has 2nd e-value at most 0.

$\boxtimes$

— P_i^{∇} $\quad P_i^{\Delta}$ — share all non-zero eigen values



P_i^{∇}

P_{i-1}^{Δ}

$$\underline{P_i^{\nabla} = U_{i-1\to i} D_{i\to i-1}}$$

$$\underline{P_{i-1}^{\Delta} = D_{i\to i-1} U_{i-1\to i}}$$

If λ is a non-zero e-value of AB then it is also a non-zero e.value of BA

$$ABv = \lambda v$$

$$BA\boxed{Bv} = B\lambda v = \lambda \boxed{Bv}.$$