Today
- Unique Games Conjecture (UGC)
  * MAXCUT
  * Vertex Cover.

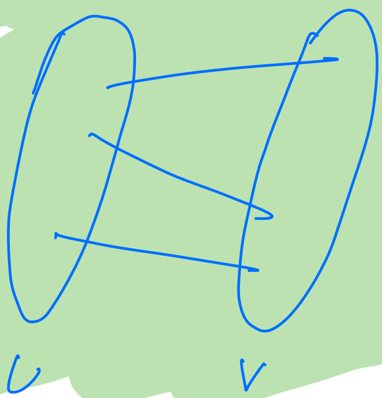
CS3.330.1 : PCPs
Limits of Approximation Algorithms
Lecture 09 (2023-4-7)
Instructor: Prahladh Harsha

Recap of where we are in the course:

SAT $\longrightarrow$ NP-completeness of combinatorial optimization

PCP Theorem
+
Parallel Repetition Theorem
/
Label Cover Theorem

Instance $\Phi = (G = (U, V, E), \underbrace{\Sigma_U, \Sigma_V}_{\text{sets of labels}}, \Psi)$



$\Psi = \{ \psi_e : \Sigma_U \times \Sigma_V \to \{0,1\} \mid e \in E \}$

Projection Cons:
$\psi_e : \Sigma_U \to \Sigma_V$

$$\mathrm{val}(\underline{\Phi}) = \max_{\substack{A: U \to \Sigma_U \\ B: V \to \Sigma_V}} \Pr_{(u,v)\sim E} \left[\Psi_{(u,v)}(A(u), B(v)) = 1 \right]$$

$0 \leq s < c \leq 1$

$\mathrm{gap}_{c,s}$-LC : Instances: Label cover instances

YES: $\{ \underline{\Phi} : \mathrm{val}(\underline{\Phi}) \geq c \}$

NO: $\{ \underline{\Phi} : \mathrm{val}(\underline{\Phi}) \leq s \}$

Thm [PCP Theorem + Parallel Repetition Theorem]

$\forall \delta \in (0,1)$, $\exists$ labels Σ_U, Σ_V, $|\Sigma_U|, |\Sigma_V| = \mathrm{poly}(\frac{1}{\delta})$ such that there is $n^{O(\log(\frac{1}{\delta}))}$ - time reduction from

SAT to $\mathrm{gap}_{1,\delta}$-LC (w/ projection constraints)

Research in inapproximability (from late 90's)

$\mathrm{gap}_{1,\delta}$-LC $\xrightarrow[\text{(Håstad)}]{}$ $\left. \begin{array}{l} \mathrm{gap}_{1,\frac{7}{8}+\delta}\text{-MAX3SAT} \\ \vdots \\ \mathrm{gap}_{1-\varepsilon,\frac{1}{2}+\delta}\text{-MAX3LIN2} \\ \vdots \end{array} \right\}$

A few problems (eg: MAXCUT, Vertex Cover, MAX2SAT) escaped obtaining "optimal" inapproximability results.

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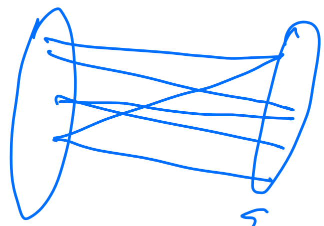
Dinur-Safra: 1.3.. -hardness for Vertex Cover.

Nature of the constraint $e=(u,v)$

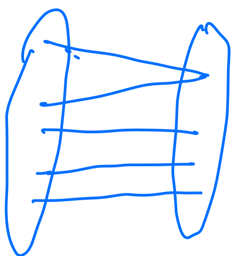
u ——— v

$\varphi_e : U \times V \to \{0,1\}$

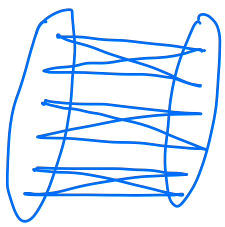


Σ_U Σ_V

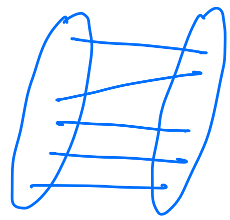
arbitrary cons



projection constraint



2-to-2



1-to-1 (unique)

LC w/ unique constraints ULC

Khot 2002:

Unique Games Conjecture:

$\forall \delta \in (0, \frac{1}{2})$, $\exists$ label Σ, $|\Sigma| = f(\delta)$

s.t. $\text{gap}_{1-\delta,\delta}$-ULC (with unique constraints) is NP-hard.

Khot:

$gap_{1-\delta,\delta}$-ULC $\xrightarrow[\text{Khot}]{}$ $\sqrt{2}$- hardness for Vertex Cover

Right after Khot 2002 paper

[Khot- Regev] $(2-\varepsilon)$- hardness for Vertex-Cover

[Khot- Kindler- Mossel - O'Donnell] Goemans Williamson alg is optimal for MAXCUT.

⋮

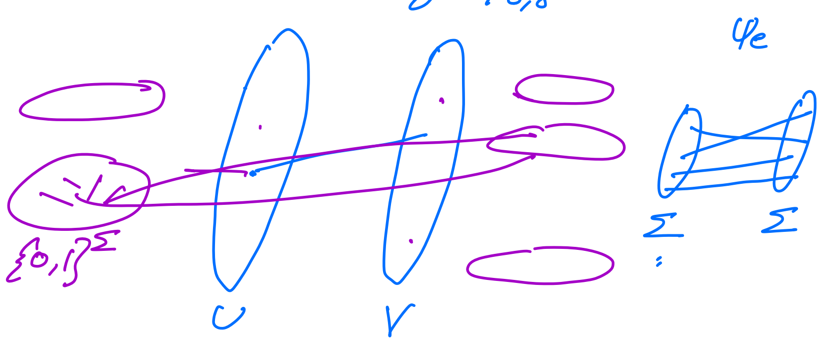
(optimal inapproximability results)

Algorithmic: Arora- Barak -Steurer.

$2^{n^{poly(\delta)}}$ - time algorithm.

UG- hardness of MAXCUT.

Redn from $gap_{1-\delta,\delta}$-ULC to MAXCUT



Cut (when localized to one of the clouds)

$f: \{0,1\}^{\Sigma} \to \{0,1\}$

Goal: Design a test that "distinguishes" Dictators from other functions.

Dictator: $f: \{0,1\}^k \to \{0,1\}$ is a dictator.

$\exists\, i \in [k]$ st $f(x) = x_i$.

$\neq$-test (first attempt)

$f: \{\pm 1\}^k \to \{\pm 1\}$

1. Pick $x \in_R \{\pm 1\}^k$
2. Query f at x & $-x$
3. Accept if $f(x) \neq f(-x)$.

This test accepts all odd functions (and is very sparse)

KKMO - suggested the following.

ρ - noise stability parameter

$\rho \in (-1, 0)$

$\neq$-KKMO-Test_ρ^f:

1. Pick $x \leftarrow_R \{\pm 1\}^k$
2. Pick $\mu \in \{\pm 1\}^k$ $\quad \mu_i \leftarrow \begin{cases} 1 & \text{w/ } \frac{1+\rho}{2} \\ -1 & \text{w/ } \frac{1-\rho}{2} \end{cases}$
3. Set $y \leftarrow x\mu$ (element-wise product)
4. Accept if $f(x) \neq f(y)$

Completeness: f is a dictator

$$\Pr[\neq_\rho\text{-test accepts } f] = \frac{1-\rho}{2} > \frac{1}{2}$$

$(\rho \in (-1,0))$.

Soundness: $f : \{\pm 1\}^k \to \{\pm 1\}$ be arbitrary.

$$\Pr[\neq_\rho\text{-test accepts}] = \Pr_{x,\mu}[f(x) \neq f(x\mu)]$$

$$= \mathop{\mathbb{E}}_{x,\mu}\left[\frac{1 - f(x)f(x\mu)}{2}\right]$$

$$= \frac{1}{2} - \frac{1}{2}\mathop{\mathbb{E}}_{x,\mu}[f(x)f(x\mu)]$$

Noise Stability: $\rho \in (-1, 1)$

$$\mathrm{Stab}_\rho(f) = \mathop{\mathbb{E}}_{\substack{x \sim \{\pm 1\}^k \\ y \sim_\rho x}} [f(x) f(y)]$$

$\mathbb{E}[y_i] = \rho x_i$

$$\mathrm{Stab}_\rho(f) = \mathbb{E}\Big[\sum_S \hat{f}(S) \chi_S(x) \sum_T \hat{f}(T) \chi_T(\mu x)\Big]$$

$$= \sum_{S,T} \hat{f}(S) \hat{f}(T) \mathop{\mathbb{E}}_x [\chi_S(x) \chi_T(x)] \cdot \mathop{\mathbb{E}}_\mu [\chi_T(\mu)]$$

$$= \sum_S \hat{f}^2(S) \rho^{|S|}$$

$\mathrm{Stab}_\rho(\text{dict}) = \rho.$

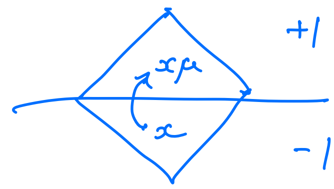
f - linear $\quad f = \prod_{i \in S} x_i \quad , |S|$

$\mathrm{Stab}_\rho(\text{linear w/ support } S) = \rho^{|S|}$

Obs:

	functions	Stability	$\Pr[\text{acc}]$
(1)	Dictators	ρ	$\frac{1-\rho}{2}$
(2)	Linear fn w/ support S	$\rho^{|S|}$	$\frac{1-\rho^{|S|}}{2}$
(3)	Constant fn	1	
(4)	Majority function. (as $k \to \infty$)	$1 - \frac{2\cos^{-1}(\rho)}{\pi}$	$\frac{\cos^{-1}(\rho)}{\pi}$

$$\cos^{-1}(x) = \frac{\pi}{2} - x - \frac{x^3}{6} + \ldots$$



<u>Thm [MOO]</u> If $f : \{\pm1\}^k \to \{\pm1\}$ satisfies $\Pr[\neq_\rho\text{-test acc}] \geq \frac{\cos^{-1}(\rho)}{\pi} + \varepsilon$ then there exists an "influential" variable.

Influence

$$\mathrm{Inf}_i(f) = \Pr_x\left[f(x) \neq f(xe_i)\right]$$

$$= \mathop{\mathbb{E}}_x\left[\frac{1 - f(x)f(xe_i)}{2}\right]$$

$$= \sum_{S \ni i} \hat{f}^2(S)$$

eg: Dictator $f = x_j$ $\qquad \mathrm{Inf}_i(f) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$

$f = \chi_S$ $\qquad \mathrm{Inf}_i(f) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{o.w.} \end{cases}$

$f = \mathrm{maj}$ $\qquad \mathrm{Inf}_i(\mathrm{maj}) = \Theta\left(\frac{1}{\sqrt{k}}\right)$

Low-order Influence:

$$\mathrm{Inf}_i^{\leq d}(f) = \sum_{\substack{S : S \ni i \\ |S| \leq d}} \hat{f}^2(S)$$

Thm [Majority is Stablest Theorem [MOO]]

For any $\rho \in (-1, 0)$, & $\varepsilon \in (0,1)$, there exists $\tau \in (0,1)$ & $d \geq 1$ s.t. $\forall\, k$, $f: \{\pm 1\}^k \to \{\pm 1\}$

if $\Pr[\neq_\rho\text{-test acc}] \geq \frac{\cos^{-1}(\rho)}{\pi} + \varepsilon$

then $\exists\ i \in [k]$, s.t. $\mathrm{Inf}_i^{\leq d}(f) \geq \tau$.

Reduction from ULC to MAXCUT

<u>KKMO-MAXCUT Test</u>$_\rho$

Input: $\Phi = (G = (U, V, E), \Sigma, \varphi)$

φ - permutation constraints

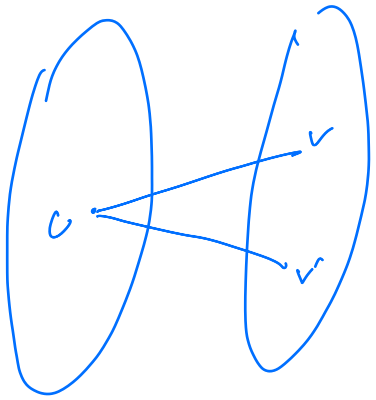
Output: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

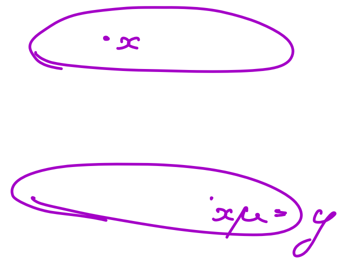
$$\mathcal{V} = V \times 2^{\Sigma}$$

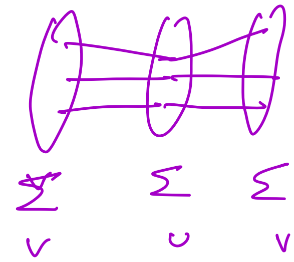
$$f_v : \{\pm 1\}^{\Sigma} \to \{\pm 1\}$$

1. Pick $u \in_R U$ w/ prob proportional to degree of u
2. Pick $v, v' \in_R N(u)$, $\pi_{(u,v)}$, $\pi_{(u,v')}$ - be the corresponding permutation constraints
3. Pick $x \leftarrow_R \{\pm 1\}^{\Sigma}$
   $y \sim_\rho x$

4. Accept iff $f_v(x \circ \pi_{(u,v)}) \neq f_{v'}(y \circ \pi_{(u,v')})$







Completeness:

Φ is ULC-instance w/ $val(\Phi) \geq 1-\delta$

$$\Downarrow$$

$\exists\, f_v : \{\pm 1\}^{\Sigma} \to \{\pm 1\}$, $\Pr[\,KKMO_\rho^f \text{ acc}\,]$

$$\geq (1-2\delta)\left(\frac{1-\rho}{2}\right)$$

$$\geq \frac{1-\rho}{2} - \varepsilon.$$

Pf: Since $val(\Phi) \geq 1-\delta$,

there exist $A : U \to \Sigma$, $B : V \to \Sigma$, $\Pr_{(u,v)}[\varphi_{(u,v)}(A(u)) = B(v)] \geq 1-\delta$

For each $v \in V$

$f_v : \{\pm 1\}^{\Sigma} \to \{\pm 1\}$

f_v = dict given by $B(v)$

Soundness Claim: $\forall \rho, \varepsilon, \exists \delta$.
Suppose $\{f_v\}_{v \in V}$ is a cut

$$\Pr[KKMO_\rho^f \text{ acc}] \geq \frac{\cos^{-1}(\rho)}{\pi} + \varepsilon$$

$$\Downarrow$$

$\exists$ a labeling A, B that satisfies more than δ fraction of ULC constraints

Cor: $\forall \rho, \varepsilon$, it is UG-hard to distinguish between MAXCUT instances
w/ — cut-size $\geq \frac{1-\rho}{2} + \varepsilon$
— cut size $\leq \frac{\cos^{-1}(\rho)}{\pi} + \varepsilon$

Proof of Soundness Claim:

$$\Pr[KKMO_\rho^f \text{ acc}] = \mathbb{E}_{u,v,v',x,\mu}\left[\frac{1 - f_v(x \circ \pi_{(u,v)}) f_{v'}(x \circ \pi_{(u,v')})}{2}\right]$$

$$\geq \frac{\cos^{-1}(\rho)}{\pi} + \varepsilon$$

By an averaging argument, for at least $\varepsilon/2$ fraction of vertices in U, we have

$$\mathop{\mathbb{E}}_{v,v',x,\mu}\left[\frac{1 - f_v(\ )f_{v'}(\ )}{2}\right] \geq \frac{\cos^{-1}(\rho)}{\pi} + \varepsilon/2.$$

$$\mathop{\mathbb{E}}_{x,\mu}\left[\frac{1 - \mathop{\mathbb{E}}_{v}[f_v(\ )]\mathop{\mathbb{E}}_{v'}[f_{v'}(\ )]}{2}\right] \geq \frac{\cos^{-1}(\rho)}{\pi} + \varepsilon/2$$

Define $g_u(x) = \mathop{\mathbb{E}}_{v \sim N(u)}\left[f_v(x \circ \pi_{(u,v)})\right]$

For every good u

$$\mathop{\mathbb{E}}_{x,\mu}\left[\frac{1 - g_u(x)g_u(x\mu)}{2}\right] \geq \frac{\cos^{-1}(\rho)}{\pi} + \varepsilon/2$$

By MIS Theorem, for each good u

there exists an influential variable

i_u, $j_u \in \Sigma$

s.t. $\mathrm{Inf}_{j_u}^{\leq d}(g_u) \geq \tau$

Define the left labelling

$A(u) = j_u$

$$g_u(x) = \mathop{\mathbb{E}}_{v \sim N(u)}\left[f_v(x \circ \pi_{(u,v)})\right]$$

$$= \mathop{\mathbb{E}}_{v}\left[\sum \hat{f}_v(S)\, \chi_S(x \circ \pi_{(u,v)})\right]$$

$$= \mathbb{E}_v \left[\sum \hat{f_v}(S)\, \chi_{\pi(u,v)(S)}(x) \right]$$

$$= \sum_T \mathbb{E}_v \left[\hat{f_v}(\pi^{-1}_{(u,v)}(T)) \right] \chi_T(x)$$

$$\hat{g_u}(T) = \mathbb{E}_v \left[\hat{f_v}(\pi^{-1}_{(u,v)}(T)) \right]$$

$$\tau \leq \mathrm{Inf}^{\leq d}_{j_u}(g_u) = \sum_{T : T \ni j_u,\ |T| \leq d} \hat{g_u}(T)^2$$

$$= \sum_{T : T \ni j_u ;\ |T| \leq d} \left(\mathbb{E}_v \left[\hat{f_v}(\pi^{-1}_{(u,v)}(T)) \right] \right)^2$$

$$\leq \sum_{T : T \ni j_u\ |T| \leq d} \mathbb{E}_v \left[\hat{f_v}^2(\pi^{-1}_{(u,v)}(T)) \right]$$

(Cauchy - Schwarz)

$$= \mathbb{E}_v \left[\mathrm{Inf}^{\leq d}_{\pi^{-1}(u,v)(j_u)}(f_v) \right]$$

Once again by an averaging argument $\tau/2$ fraction of nbrs of u satisfy

$$\mathrm{Inf}^{\leq d}_{\pi^{-1}(u,v)(j_u)}(f_v) \geq \tau/2.$$

$$S_v = \{ j \in \Sigma \mid \mathrm{Inf}_j^{\leq d}(f_v) \geq \tau/2 \}$$

$$|S_v| \leq 2d/\tau \quad \left(\text{since } \sum_i \mathrm{Inf}_i^{\leq d}(f) \leq d\right)$$

$B : V \to \Sigma$

Pick a random $\sigma \leftarrow S_v$

$$\Pr_{u,v,B}\left[\varphi_{(u,v)}(A(u)) = B(v)\right] \geq \frac{\varepsilon}{2} \cdot \frac{\tau}{2} \cdot \frac{\tau}{2d}$$

$$\geq \frac{\varepsilon \tau^2}{8d}$$

Choose $\delta < \varepsilon\tau^2/8d$.

$\boxtimes$.