Algebraic Circuit Complexity

Problem Set 2

Due date: October 15^{th}, 2017

Instructions

1. The problem set has **6 questions** with a total score of **100 points**.
2. You are expected to work independently.
3. Solutions are expected as a LaTeX document.
4. The deadline is 15th October 2017. You can also submit answers to some (or all) of the questions **any time after the deadline** (a little before the course ends, of course; they need to be graded) for **half the credit**.

   This is to encourage you to solve all the question in these problem sets, even if it is past the deadline.

Questions

Question 1. (10 points) *Recall that* Perm_n *is the permanent of a symbolic* $n \times n$ *matrix with distinct variables for each of its entries. If* $\mathbb{F}_q$ *is the finite field of* q *elements, then prove that*

$$\Pr_{\mathbf{a} \in_R \mathbb{F}_q^{n^2}} [\mathrm{Perm}_n(\mathbf{a}) \neq 0] \geq \frac{q-2}{q-1}.$$

Even if you can show some non-trivial lower bound for the probability purely as a function of q *that would be fine.*

*(**Hint:** Koutis + induction)*

Question 2. (10 points) *For a circuit* C *(with arbitrary fan-ins), we shall say that the* product-depth *of the circuit is* Δ *if any path from root to leaf passes through at most* Δ *multiplication gates. For instance, a* $\Sigma\Pi\Sigma\Pi\Sigma$ *circuit has product-depth* 2.

Show that if $f(x_1, \ldots, x_n)$ *is a polynomial of degree* d *that can be computed by an algebraic circuit of size* s*, then for any* Δ*, it can also be computed by an algebraic circuit of size* $s^{O(d^{1/\Delta})}$ *and product-depth* Δ*.*

Question 3. (20 points) *Prove a super-polynomial lower bound for $\Sigma\Pi^{[n^{1.5}]}\Sigma$ circuits (that is, depth-3 circuits whose formal degree is bounded by $n^{1.5}$) computing* Det_n.

Actually, any $n^{2-\delta}$ would also be possible but I just put a specific constant less than 2.

Recall that $\binom{n}{\alpha n} \approx 2^{H(\alpha)n}$, and $H(\alpha) \approx \alpha \log(1/\alpha)$ when α is small.

Question 4. (20 points) *Consider circuits of the following kind:*

$$C = \sum_{i=1}^{s} H_i(\ell_{i,1}, \ldots, \ell_{i,r})$$

- $r = 100$,
- *each $\ell_{i,j}$ is a linear polynomial,*
- *each $H_i(z_1, \ldots, z_r) \in \mathbb{F}[z_1, \ldots, z_r]$ is an arbitrary polynomial on $r = 100$ variables.*

No guarantees on the degree of H_i. Could be HUGE, but somehow manages to cancel high-degree terms eventually

Find an explicit polynomial $f(x_1, \ldots, x_n)$ such that if the above circuit is to compute f, then s (the top fan-in) must be exponentially large.

Question 5. (25 points) *Recall the definition of the* iterated matrix multiplication *polynomial* $(\mathrm{IMM}_{n,d})$

$$\mathrm{IMM}_{n,d} = \left(\begin{bmatrix} x_{11}^{(1)} & \cdots & x_{1n}^{(1)} \\ \vdots & \ddots & \vdots \\ x_{n1}^{(1)} & \cdots & x_{nn}^{(1)} \end{bmatrix} \cdots \begin{bmatrix} x_{11}^{(d)} & \cdots & x_{1n}^{(d)} \\ \vdots & \ddots & \vdots \\ x_{n1}^{(d)} & \cdots & x_{nn}^{(d)} \end{bmatrix} \right)_{1,1}$$

where each $x_{ij}^{(k)}$ is a distinct variable. This is a homogeneous polynomial of degree d over $\approx n^2 d$ variables. For any $0 \leq k \leq d/4$, show that

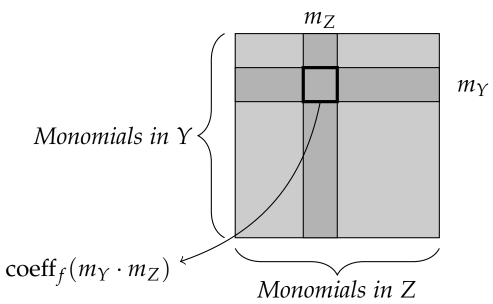
Depends on $(d-2)n^2 + 2n$ variables actually.

$$\dim \left(\partial^{=k}(\mathrm{IMM}_{n,d}) \right) \geq n^k.$$

Hint: *Think of $\mathrm{IMM}_{n,d}$ as the canonical width-n ABP where each edge gets a fresh variable as weight. What do derivatives with respect to k-edges mean in the ABP computation? Consider taking derivatives only with respect to layers $2, 6, 10, 14, \ldots, 4k-2$ and set many of the variables in the other layers of $\mathrm{IMM}_{n,d}$ to zero. Do this cleverly so that each such derivative yields a distinct monomial.*

Question 6. (15 points) *In class, we saw Nisan's partial derivative matrix specifically for multilinear polynomials but it can also be defined for non-multilinear polynomials. For any partition of the variables in to $X = Y \sqcup Z$, define the matrix $M_{Y,Z}(f)$*

$M_{Y,Z}(f) \quad =$



The rows and columns would now also involve non-multilinear monomials in Y and Z respectively.

If $X = Y \sqcup Z$ is a partition with $|Y| = |Z| = n/2$, what is $\text{rank}(M_{Y,Z}(f))$ *for the following polynomials?*

You don't need to compute it exactly but you should get a rough estimate (is it $O(1)$, or $\text{poly}(n, d)$ or $\exp(d)$, etc.).

1. $f(X) = (x_1 + \cdots + x_n)^d$,
2. $f(X) = \text{ESym}_d$,
3. $f(X) = h_d(x_1, \ldots, x_n)$, *the complete homogeneous symmetric polynomial of degree d.*

We used the polynomial h_d in the last problem set as well.